EXAMPLE 2.17 Use the PA = LU factorization to solve the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}.$$

$PA = LU \Rightarrow PAx = Pb$

P_{231}

$LUx = Pb$

C

$LC = Pb$

Solve for C → Use $Ux = C$ solve for x

► EXAMPLE 2.17 Use the PA = LU factorization to solve the system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}.$$

P_{23},

$$L \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \quad C \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$

3×3 $\quad 3 \times 1$

$$\begin{bmatrix} C_1 \\ \frac{1}{4}C_1 + C_2 + 0 \\ \frac{1}{2}C_1 - \frac{1}{2}C_2 + C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}$$

$\Rightarrow C_1 = 0$

$\Rightarrow C_2 = 6$

$0 - 3 + C_3 = 5$

$C_3 = 8$

$$C = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix} \Rightarrow UX = C$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix}$$

$8x_3 = 8 \Rightarrow x_3 = 1$

Back substitute $\Rightarrow 2x_2 + 2x_3 = 6$

$x_2 = \frac{1}{2}(6 - 2(1)) = 2$

Back substitute $4x_1 + 4x_2 - 4x_3 = 0$

$4x_1 = (4)(1) - 4(2) \Rightarrow x_1 = \frac{1}{4}(4 - 8) = -1$

$$Ax = b$$

Check $\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+2+5 \\ -4+8-4 \\ -1+6+1 \end{bmatrix}$

b

$\begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$

$$X_1 = -1$$
$$X_2 = 2$$
$$X_3 = 1$$

EXAMPLE 2.17 Use the PA = LU factorization to solve the system $Ax = b$, where

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$PA = LU \Rightarrow Ax = b$

other alternative

$(PA)x = Pb$

$LUx = Pb$

$Ix = U^{-1}L^{-1}Pb$

$(U^{-1}L^{-1})(LU)x = U^{-1}L^{-1}Pb$

$\Rightarrow x = U^{-1}L^{-1}Pb$

P231

$[U|I] \Rightarrow [I|U^{-1}]$ Find U^{-1}

$$\left[\begin{array}{ccc|ccc} 4 & 4 & -4 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \to R_1/4 \\ \to R_2/2 \\ \to \frac{R_3}{8} \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{8} \end{array}\right]$$

$$\to R_3 + R_1/R_1 = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{8} \end{array}\right] \begin{array}{l} \to -\frac{R_2 + R_1}{R_1} \\ \to R_2 \\ \to R_3 \end{array}$$

$$\to -\frac{R_3 + R_2}{R_2}$$

$$\to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{8} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{8} \end{array}\right]$$

Check $\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$

$U \qquad\qquad U^{-1}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[L \mid I] \Rightarrow [I \mid L^{-1}]$

Find L^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow -\frac{R_1}{4} + R_2 \rightarrow -\frac{R_1}{2} + R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$\rightarrow R_1$

$\rightarrow R_2$

$\rightarrow +\frac{1}{2}R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & 1 \end{array}\right]$$

Check $L L^{-1} = I$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{5}{8} & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$U^{-1} L^{-1} P_{231} b = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{8} \\ 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{5}{8} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{7}{32} & -\frac{3}{8} & \frac{1}{4} \\ -\frac{3}{64} & \frac{7}{16} & -\frac{1}{8} \\ -\frac{5}{64} & \frac{1}{16} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$x_1 = -1$
$x_2 = 2$
$x_3 = 1$

Same