Subject: Partial Pivoting For 3x3 Matrix

(a) Solved Problems 2.16

Find L U Matrix For PA

(b) Solved Problem 2.17

For the Same A Solve For onknowns

## 2.4.3 PA = LU factorization

In this section, we put together everything we know about Gaussian elimination into the PA = LU factorization. This is the matrix formulation of elimination with partial pivoting. The PA = LU factorization is the established workhorse for solving systems of linear equations.

As its name implies, the PA = LU factorization is simply the *LU* factorization of a row-exchanged version of *A*. Under partial pivoting, the rows that need exchanging are not known at the outset, so we must be careful about fitting the row exchange information into the factorization. In particular, we need to keep track of previous multipliers when a row exchange is made. We begin with an example.






# Partial Pivoting

**EXAMPLE 2.16** Find the PA = LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} \textcircled{2} & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Select $\max(4, 1) = 4$

shift $r_2 \rightleftarrows r_1$

$P_A$

$$\begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

Eliminate $a_{21}$, $a_{31}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & ? & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \Rightarrow U_1$$

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & ? & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}r_1 + r_2} \xrightarrow{-\frac{1}{4}r_1 + r_3}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & \\ \frac{1}{2} & L_{32} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & -1 & 7 \\ 0 & 2 & 2 \end{bmatrix}$$

select $\max(|-1|, |2|) \rightarrow 2$

swap $r_3 \Rightarrow r_2$

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & -1 & 7 \\ 0 & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & -1 & 7 \end{bmatrix} \xrightarrow{+\frac{1}{2}R_2 + R_3} \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow final\ U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \Rightarrow final\ L$$

$$L_{32} = -\frac{1}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & L_{32} & 1 \end{bmatrix} \quad L_1$$

$(P_{132} P_{213} A) = (L \quad U)$ $\quad L \quad U$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -4 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

L.H.S

R.H.S

This is the finished elimination. Now we can read off the PA = LU factorization:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{A}=\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}_{U} \qquad (2.22)$$

$P_{231}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$P_{132}$ $P_{213}$

$P_{231}$