2.4.3 PA = LU factorization

In this section, we put together everything we know about Gaussian elimination into the PA = LU factorization. This is the matrix formulation of elimination with partial pivoting. The PA = LU factorization is the established workhorse for solving systems of linear equations.

As its name implies, the PA = LU factorization is simply the *LU* factorization of a row-exchanged version of *A*. Under partial pivoting, the rows that need exchanging are not known at the outset, so we must be careful about fitting the row exchange information into the factorization. In particular, we need to keep track of previous multipliers when a row exchange is made. We begin with an example.





EXAMPLE 2.14 Apply Gaussian elimination with partial pivoting to solve the system (2.1).

The equations can be written in tableau form as

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right]$$

solution

$\Rightarrow x_1 = 2$

$x_2 = 1$

$x_1 + x_2 = 3$ I

$3x_1 - 4x_2 = 2$ II

$A \mid b$

$3 > 1$ replace

Use absolute max $= 3$

$$\left[\begin{array}{cc} 1 & 1 \\ 3 & -4 \end{array}\right]$$

Swap $r_1 \to r_2$

$$\hookrightarrow \left[\begin{array}{cc} 3 & -4 \\ 1 & 1 \end{array}\right] \begin{array}{l} \to R_1 \\ \to -\frac{1}{3}R_1 + R_2 \\ \quad\; R_2 \end{array}$$

$$\begin{matrix} & A & & b \\ \left[\begin{matrix} 1 & 1 \\ 3 & -4 \end{matrix}\right. & \Big| & \left.\begin{matrix} 3 \\ 2 \end{matrix}\right] \end{matrix} \Rightarrow \text{after swap} \Rightarrow \begin{matrix} & PA & & Pb \\ \left[\begin{matrix} 3 & -4 \\ 1 & 1 \end{matrix}\right. & \Big| & \left.\begin{matrix} 2 \\ 3 \end{matrix}\right] \end{matrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow -\frac{R_1}{3} + R_2 \end{matrix}$$

$$\begin{matrix} & x_1 & x_2 & \\ \left[\begin{matrix} 3 & -4 \\ 0 & 7/3 \end{matrix}\right. & \Big| & \left.\begin{matrix} 2 \\ 7/3 \end{matrix}\right] \end{matrix}$$

$$x_2 = \frac{7}{3} / 7/3 = 1$$

Back Substitute

$$3x_1 - 4(x_2) = 2$$

$$3x_1 = 2 + 4(1) = 6$$

$$x_1 = 2$$

$$x_1 = 2$$

$$x_2 = 1$$

$A \Rightarrow PA$

PA

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \Rightarrow PA$$

U

$$\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \rightarrow R_1$$

$$\rightarrow -\frac{1}{3}R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 3 & -4 \\ 0 & \frac{7}{3} \end{bmatrix}$$

check LU product

$L_{21} = -(-\frac{1}{3})$

L $\quad U \quad \Rightarrow PA$

$$\begin{bmatrix} 1 & 0 \\ +\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 0 & \frac{7}{3} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}$$

$$(PA)X = Pb \Rightarrow (LU)X = Pb$$

PA

$$AX = b$$

check that $(L\ U)X = PA$ $\quad L \quad U \quad P_{21}b$

$$P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & 7/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$3x_1 - 4x_2 = 2$$

$$x_1 + x_2 = 3$$

$$(LU)x = \underset{PA}{P_{21}b} \Rightarrow U^{-1}L^{-1}(LU)x = U^{-1}L^{-1}P_{21}b$$

Find L^{-1} & U^{-1} To get (x, y)

$$L = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix} \Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} /1$$

$$U = \begin{bmatrix} 3 & -4 \\ 0 & 7/3 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} 7/3 & 4 \\ 0 & 3 \end{bmatrix} /7$$

$$X = (U^{-1} L^{-1}) P_{21} (b)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7/3 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad b$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 14 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x_1 = 2, \quad x_2 = 1$