

First Solved problem

Find L & U values using option - 2

Turn $A \rightarrow$ upper matrix - U

[Use row operation
Use Elementary matrices

Find inverse of L and check

$$LU = A \quad \& \quad L^{-1}L = I$$

First Example

Write the lower and upper matrix for

Crowt's method $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

Option - 2

Solution: Convert $A \rightarrow$ an upper matrix

$|A| = +10$ Why?

$$\begin{bmatrix} 1 & ? & ? \\ & 1 & ? \\ & & 1 \end{bmatrix}$$

Check if matrix is invertible, estimate the determinant value.

$$\begin{aligned} |A| &= +1 \begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix} - 1 \begin{vmatrix} 4 & -1 \\ 3 & 3 \end{vmatrix} + 1 \begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} \\ &= +1(9+5) - 1(12+3) + (20-9) \\ &= +14 - 15 + 11 \\ &= +10 \end{aligned}$$

Prepared by Eng. Maged Kamel.

$$A = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \cdot \begin{bmatrix} +1 & U_{12} & U_{13} \\ 0 & +1 & U_{23} \\ 0 & 0 & +1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21} & L_{21}U_{12} + L_{22} & L_{21}U_{13} + L_{22}U_{23} \\ L_{31} & L_{31}U_{12} + L_{32} & L_{31}U_{13} + L_{32}U_{23} + L_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$L_{11} = a_{11} = 1$$

$$L_{21} = a_{21} = 4$$

$$L_{31} = a_{31} = 3$$

Compare the first column of A and LU

Prepared by Eng. Maged Kamel.

Compare the first row of A and LU

$$\begin{aligned} L_{11} &= 1 \\ L_{11} U_{12} &= 1 \quad / \quad L_{11} = U_{12} = \frac{1}{1} = +1 \\ L_{11} U_{13} &= 1 \quad / \quad U_{13} = \frac{1}{1} = +1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} L_{11} & L_{11} U_{12} & L_{11} U_{13} \\ L_{21} & L_{21} U_{12} + L_{22} & L_{21} U_{13} + L_{22} U_{23} \\ L_{31} & L_{31} U_{12} + L_{32} & L_{31} U_{13} + L_{32} U_{23} + L_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} u_{12} & l_{11} u_{13} \\ l_{21} & l_{21} u_{12} + l_{22} & l_{21} u_{13} + l_{22} u_{23} \\ l_{31} & l_{31} u_{12} + l_{32} & l_{31} u_{13} + l_{32} u_{23} + l_{33} \end{bmatrix}$$

By definition

$$\begin{cases} U_{11} = U_{22} = U_{33} = 1 \\ L_{11} = a_{11} = 1 \\ L_{21} = a_{21} = 4 \\ L_{31} = a_{31} = 3 \end{cases}$$

Prepared by Eng. Maged Kamel.

Check what do we have ?

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & ? & 0 \\ 3 & ? & ? \end{bmatrix} \quad \begin{matrix} L_{22} ? \\ L_{33} ? \end{matrix} \quad \& \quad L_{32} = ?$$

$$\begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow U_{23} ??$$

Prepared by Eng. Maged Kamel.

Find L22 and L32

Elementary matrix E_1

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

A

$\Rightarrow$ Use a_{11} as a pivot

$$-\frac{a_{21}}{a_{11}} = -\frac{4}{1} = -4 \rightarrow \text{multiply by } \frac{R_1 + R_2}{R_2}$$

$$-\frac{a_{31}}{a_{11}} = -\frac{3}{1} = -3 \rightarrow \text{multiply by } \frac{R_1 + R_3}{R_3}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & +2 & 0 \end{bmatrix}$$

A_1

$$\Rightarrow \begin{bmatrix} L_{11} & L_{11}u_{12} & L_{11}u_{13} \\ 0 & L_{22} & L_{22}u_{23} \\ 0 & L_{32} & L_{32}u_{23} + L_{33} \end{bmatrix}$$

$L_{22} = -1$
 $L_{32} = +2$

same

Prepared by Eng. Maged Kamel.

Find U23

$\Rightarrow$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Elementary matrix E2

A_1

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & +2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & L_{22} & L_{22}U_{23} \\ 0 & L_{32} & L_{32}U_{23} + L_{33} \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{R_2}$

$L_{22} = -1$
 $L_{32} = +2$

same

$$\left(\frac{a_{23}}{a_{22}} \right)_{A_1} = \frac{L_{22} U_{23}}{L_{22}} = U_{23} = -5 / -1 = +5$$

U matrix is complete $\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Find L33

$\Rightarrow$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$L_{11} = 1 \quad \& \quad L_{31} = 3$$

$$L_{22} = 1$$

$A1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & +2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} L_{11} & L_{11}u_{12} & L_{11}u_{13} \\ 0 & L_{22} & L_{22}u_{23} \\ 0 & L_{32} & L_{32}u_{23} + L_{33} \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{R_2}$

$\swarrow$ same $\nearrow$

Use L_{22} as a pivot $\Rightarrow \frac{L_{32}}{L_{22}}(-1) = \left(\frac{2}{-1}\right)(-1) = +2$

$-2 R_2 + R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10+0 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{11}u_{12} & L_{11}u_{13} \\ 0 & L_{22} & L_{22}u_{23} \\ 0 & 0 & L_{33} \end{bmatrix}$$

To get U matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} L_{11} & L_{11} U_{12} & L_{11} U_{13} \\ 0 & L_{22} & L_{22} U_{23} \\ 0 & 0 & L_{33} \end{bmatrix}$$

$$\begin{aligned} 1/L_{11} &\Rightarrow [1/1] \\ 1/L_{22} &\Rightarrow [1/-1] \\ 1/L_{33} &\hookrightarrow [1/-10] \end{aligned}$$

For Elementary operation E_3

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & +1 & +5 \\ 0 & 0 & +1 \end{bmatrix}$$

$\Rightarrow$ U matrix

$$L_{33} = -10 \Rightarrow L \text{ matrix is Complete}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Check $L U \Rightarrow A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \Rightarrow \text{Same given } A$$