

Crout's Method

Consider the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above system can be written as

$$AX = B$$

Let

Lower matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\text{and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Upper Matrix

...(2)

...(3)

with diagonal of 1's

where

$$L = \begin{bmatrix} l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} l_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

Case of three systems of linear equations

...(1)

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating the corresponding elements, we get

$$l_{11} = a_{11} \quad l_{21} = a_{21} \quad l_{31} = a_{31} \quad \dots(i)$$

$$l_{11}u_{12} = a_{12} \quad l_{11}u_{13} = a_{13} \quad \dots(ii)$$

$$l_{21}u_{12} + l_{22} = a_{22} \quad l_{31}u_{12} + l_{32} = a_{32} \quad \dots(iii)$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \quad \dots(iv)$$

$$\text{and} \quad l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33} \quad \dots(v)$$

$$\text{from (ii) we get} \quad u_{12} = a_{12}/l_{11} \quad \text{(using (i))}$$

$$= a_{12}/a_{11}$$

$$\text{from (iii) we get} \quad l_{22} = a_{22} - l_{21}u_{12} \quad \dots(vi)$$

$$l_{32} = a_{32} - l_{31}u_{12} \quad \dots(vii)$$

$$(iv) \text{ gives} \quad u_{23} = (a_{23} - l_{21}u_{13})/l_{22} \quad \dots(viii)$$

from the relation (v) we get

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \quad \dots(ix)$$

strategy to get $L \& U$

Divide $\Rightarrow U_{12}$
 $L_{11} \rightarrow a_{11}$

Divide by a_{11}

Divide by L_{11}

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Copy and place

$$\begin{bmatrix} =a_{11} & a_{12} & a_{13} \\ =a_{21} & * & * \\ =a_{31} & * & * \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ a_{21} & * & * \\ a_{31} & * & * \end{bmatrix} [L_{11} \quad U_{12} \quad U_{13}]$$

$$\left. \begin{array}{l} a_{11} : l_{11} \\ a_{21} : l_{21} \\ a_{31} : l_{31} \end{array} \right\}$$

A-matrix

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A$$

option-1

$$\left. \begin{aligned} L_{11} &= a_{11} \\ L_{21} &= a_{21} \\ L_{31} &= a_{31} \end{aligned} \right\} \begin{aligned} U_{12} &= \frac{a_{12}}{a_{11}} \\ U_{13} &= \frac{a_{13}}{a_{11}} \end{aligned}$$

Convert Matrix A into Lower matrix

- (a) The First Column $\left. \begin{aligned} a_{11} &= L_{11} \\ a_{12} &= L_{21} \\ a_{13} &= L_{31} \end{aligned} \right\} \Rightarrow \text{Turn } A \rightarrow \text{Lower matrix}$
- (b) Use a_{11} as a pivot

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} - \frac{a_{12}}{a_{11}}(a_{21}) & a_{23} - \frac{a_{13}}{a_{11}}(a_{21}) \\ a_{32} - \frac{a_{12}}{a_{11}}(a_{31}) & a_{33} - \frac{a_{13}}{a_{11}}(a_{31}) \end{bmatrix}$$

$$\begin{aligned} & -\frac{a_{12}}{a_{11}} C_1 + C_2 \quad \downarrow \text{2nd Column} \\ & -\frac{a_{13}}{a_{11}} C_1 + C_3 \quad \downarrow \text{3rd Column} \end{aligned}$$

From the previous slide

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Divide

$$-\frac{L_{11}u_{12}}{L_{11}}(C_1) + C_2$$

$$-\frac{L_{11}u_{13}}{L_{11}}C_1 + C_3$$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{21}u_{12} + L_{22} - u_{12}L_{21} & L_{21}u_{13} + L_{22}u_{23} - u_{13}L_{21} \\ L_{31} & L_{31}u_{12} + L_{32} - u_{12}L_{31} & L_{31}u_{13} + L_{32}u_{23} + L_{33} - u_{13}L_{31} \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & L_{22}u_{23} \\ L_{31} & L_{32} & L_{32}u_{23} + L_{33} \end{bmatrix}$$

Similar items were deleted

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & L_{22} U_{23} \\ L_{31} & L_{32} & L_{32} U_{23} + L_{33} \end{bmatrix} \quad \text{Use } L_{22} \text{ as a pivot}$$

⇓ Corresponding a_{ij}

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} - \frac{a_{12}}{a_{11}} a_{21} & a_{23} - \frac{a_{13}}{a_{11}} a_{21} \\ a_{31} & a_{32} - \frac{a_{12}}{a_{11}} a_{31} & a_{33} - \frac{a_{13}}{a_{11}} a_{31} \end{bmatrix}$$

Use L_{22} as a pivot Divide $\frac{a_{11}}{L_{22}} U_{23}$
 $-\frac{L_{22} U_{23}}{L_{22}} C_2 + C_3 \rightarrow$ to eliminate $L_{22} U_{23}$
 $\& L_{32} U_{23}$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & \cancel{L_{22} U_{23}} - \frac{\cancel{L_{22} U_{23}} (\cancel{L_{22}})}{L_{22}} \\ L_{31} & L_{32} & \cancel{L_{32} U_{23}} - \frac{\cancel{L_{22} U_{23}} (\cancel{L_{32}})}{L_{22}} + L_{33} \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

Final Lower
Matrix

Steps

$$\begin{matrix} & \leftarrow A \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{matrix}$$

C_1 = First Column

C_2 = 2nd Column

Multiply $-\frac{a_{12}}{a_{11}}a_{11} + C_2$

$-\frac{a_{13}}{a_{11}}a_{11} + C_3$

Consider

Convert $A \rightarrow$ Lower matrix

(a) Use a_{11} as a pivot

(b) Use a_{11} as a pivot

(c) get L_{22} value & $L_{32} \rightarrow a_{22}, a_{32}$ &

(d) Divide a_{32} element $\left(-\frac{a_{32}}{a_{22}} C_2 \right) \Rightarrow$ add to C_3

get L_{33} value as new a_{33} element

$$L_{11} : a_{11}$$

$$L_{21} : a_{21}$$

$$L_{31} : a_{31}$$

U Element

$$U_{12} = \frac{a_{21}}{a_{11}}$$

$$U_{13} = \frac{a_{31}}{a_{11}}$$

$$U_{23} = \frac{a_{23}(\text{new})}{a_{22}}$$

$$U_{11} = 1$$

$$U_{22} = 1$$

$$U_{23} = 1$$