$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & L_{22}U_{23} \\ L_{31} & L_{32} & L_{32}U_{23} + L_{33} \end{bmatrix}$$

Use L_{22} as a pivot

$\Downarrow$ Corresponding a_{ij}

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} - \frac{a_{12}}{a_{11}} a_{21} & a_{23} - \frac{a_{13}}{a_{11}} a_{21} \\ a_{31} & a_{32} - \frac{a_{12}}{a_{11}} a_{31} & a_{33} - \frac{a_{13}}{a_{11}} a_{31} \end{bmatrix}$$

Use L_{22} as a pivot Divide $-\frac{L_{22}U_{23}}{L_{22}}$

$-\frac{L_{22}U_{23}}{L_{22}} C_2 + C_3 \rightarrow$ to Eliminate $L_{22}U_{23}$ & $L_{32}U_{23}$

$$
\begin{bmatrix}
L_{11} & 0 & 0 \\
L_{21} & L_{22} & \cancel{L_{22}U_{23}} - \frac{\cancel{L_{22}U_{23}}}{L_{22}}(\cancel{L_{22}}) \\
L_{31} & L_{32} & \cancel{L_{32}}U_{23} - \frac{\cancel{L_{22}U_{23}}}{L_{22}}(L_{32}) + L_{33}
\end{bmatrix}
$$

$$
\begin{bmatrix}
L_{11} & 0 & 0 \\
L_{21} & L_{22} & 0 \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
$$

Final Lower Matrix

Steps

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiply $-\frac{a_{12}}{a_{11}} a_{11} + C_2$

C_1 : First Column

C_2 : 2nd Column $-\frac{a_{13}}{a_{11}} a_{11} + C_3$

Convert $A \rightarrow$ Lower matrix

Consider

$L_{11} : a_{11}$

$L_{21} : a_{21}$

$L_{31} : a_{31}$

(a) Use a_{11} as a pivot $-\frac{a_{12}}{a_{11}} C_1 + C_2$

(b) Use a_{11} as a pivot $-\frac{a_{13}}{a_{11}} C_1 + C_3$

(c) get L_{22} value & $L_{32} \rightarrow a_{22}, a_{32}$ E

(d) Divide a_{32} Element $\left(-\frac{a_{32}}{a_{22}} C_2\right) \Rightarrow$ add to C_3

get L_{33} value as new a_{33} Element

U Element

$$\left.\begin{aligned} U_{12} &= \frac{a_{21}}{a_{11}} \\ U_{13} &= \frac{a_{31}}{a_{11}} \end{aligned}\right\} \quad \begin{aligned} U_{11} &= 1 \\ U_{22} &= 1 \\ U_{23} &= 1 \end{aligned}$$

$$U_{23} = \frac{a_{23}}{a_{22}} \text{(new)}$$

Crout's Method

Case of three systems of Linear. Equations

Consider the system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
\qquad \ldots(1)
$$

The above system can be written as

$$AX = B \qquad \ldots(2)$$

upper Matrix

Let Lower matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots(3)$$

with diagonal of 1's

where

$$L = \begin{bmatrix} l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

$$\Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating the corresponding elements, we get

$$l_{11} = a_{11} \quad l_{21} = a_{21} \quad l_{31} = a_{31} \qquad \text{...}(i)$$

$$l_{11}u_{12} = a_{12} \quad l_{11}u_{13} = a_{13} \qquad \text{...}(ii)$$

$$l_{21}\ u_{12} + l_{22} = a_{22} \quad l_{31}\ u_{12} + l_{32} = a_{32} \qquad \text{...}(iii)$$

$$l_{21}\ u_{13} + l_{22}\ u_{23} = a_{23} \qquad \text{...}(iv)$$

and $$l_{31}\ u_{13} + l_{32}\ u_{23} + l_{33} = a_{33} \qquad \text{...}(v)$$

from (*ii*) we get $$u_{12} = a_{12}/l_{11} \qquad \text{(using } (i))$$
$$= a_{12}/a_{11}$$

from (*iii*) we get $$l_{22} = a_{22} - l_{21}\ u_{12} \qquad \text{...}(vi)$$

$$l_{32} = a_{32} - l_{31}\ u_{12} \qquad \text{...}(vii)$$

(*iv*) gives $$u_{23} = (a_{23} - l_{21}\ u_{23})/l_{22} \qquad \text{...}(viii)$$

from the relation (*v*) we get

$$l_{33} = a_{33} - l_{31}\ u_{13} - l_{32}\ u_{23} \qquad \text{...}(ix)$$

Prepared by Eng.Maged Kamel.

Strategy to get L & U

Divide / L_{11} → a_{11} → U_{12}

Divide by / a_{11}

Divide by / L_{11}

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

Copy and place

$$\begin{bmatrix} =a_{11} & a_{12} & a_{13} \\ =a_{21} & * & * \\ =a_{31} & * & * \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ a_{21} & * & * \\ a_{31} & * & * \end{bmatrix} \Rightarrow \begin{bmatrix} L_{11} & U_{12} & U_{13} \end{bmatrix}$$

$a_{11} : L_{11}$

$a_{21} : L_{21}$

$a_{31} : L_{31}$

A-matrix

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix} \Rightarrow \overset{A}{\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}$$

option-1

$L_{11} = a_{11}$ $\quad U_{12} : \frac{a_{12}}{a_{11}}$

$L_{21} = a_{21}$

$L_{31} = a_{31}$ $\quad U_{13} : \frac{a_{13}}{a_{11}}$

Convert Matrix A into Lower matrix

(a) The First Column $\left.\begin{matrix} a_{11} : L_{11} \\ a_{12} : L_{21} \\ a_{13} : L_{31} \end{matrix}\right\}$ $\Rightarrow$ Turn $A \to$ Lower matrix

(b) Use a_{11} as a pivot

$-\frac{a_{12}}{a_{11}} C_1 + C_2$ $\quad -\frac{a_{13}}{a_{11}} C_1 + C_3$

2nd Column $\quad$ First Column $\quad$ 3rd Column

$$\begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}} & \begin{matrix} 0 \\ a_{22} - \frac{a_{12}}{a_{11}}(a_{21}) \\ a_{32} - \frac{a_{12}}{a_{11}} \cdot a_{31} \end{matrix} & \begin{matrix} 0 \\ a_{23} - \frac{a_{13}}{a_{11}}(a_{21}) \\ a_{33} - \frac{a_{13}}{a_{11}} \cdot (a_{31}) \end{matrix} \end{bmatrix}$$

From the Previous slide

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Divide

$-\frac{L_{11}U_{12}}{L_{11}}(C_1)+C_2$

$-\frac{L_{11}U_{13}}{L_{11}}C_1+C_3$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & \cancel{L_{21}U_{12}} + L_{22} - \cancel{U_{12}L_{21}} & \cancel{L_{21}U_{13}} + L_{22}U_{23} - \cancel{U_{13}L_{21}} \\ L_{31} & \cancel{L_{31}U_{12}} + L_{32} - \cancel{U_{12}L_{31}} & \cancel{L_{31}U_{13}} + L_{32}U_{23} + L_{33} - \cancel{U_{13}L_{31}} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & L_{22}U_{23} \\ L_{31} & L_{32} & L_{32}U_{23} + L_{33} \end{bmatrix}$$

similar items were deleted