Crout's Method

option - 2

$A \rightarrow U$

get L_{22}

$L_{32} \rightarrow L_{33}$

Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \text{...(1)}$$

The above system can be written as

$$AX = B \qquad \text{...(2)}$$

Let

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{...(3)}$$

where

$$L = \begin{bmatrix} l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} l_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating the corresponding elements, we get

$$l_{11} = a_{11} \qquad l_{21} = a_{21} \qquad l_{31} = a_{31} \qquad \ldots(i)$$

$$l_{11}u_{12} = a_{12} \qquad l_{11}u_{13} = a_{13} \qquad \ldots(ii)$$

$$l_{21}\ u_{12} + l_{22} = a_{22} \qquad l_{31}\ u_{12} + l_{32} = a_{32} \qquad \ldots(iii)$$

$$l_{21}\ u_{13} + l_{22}\ u_{23} = a_{23} \qquad \ldots(iv)$$

and $$l_{31}\ u_{13} + l_{32}\ u_{23} + l_{33} = a_{33} \qquad \ldots(v)$$

from (*ii*) we get $$u_{12} = a_{12}/l_{11} \qquad \text{(using } (i))$$

$$= a_{12}/a_{11}$$

from (*iii*) we get $$l_{22} = a_{22} - l_{21}\ u_{12} \qquad \ldots(vi)$$

$$l_{32} = a_{32} - l_{31}\ u_{12} \qquad \ldots(vii)$$

(*iv*) gives $$u_{23} = (a_{23} - l_{21}\ u_{23})/l_{22} \qquad \ldots(viii)$$

from the relation (*v*) we get

$$l_{33} = a_{33} - l_{31}\ u_{13} - l_{32}\ u_{23} \qquad \ldots(ix)$$

Strategy to get L & U

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Divide / L_{11} → U_{12}

Divide by a_{11}

Copy and place

$$\begin{bmatrix} = a_{11} & a_{12} & a_{13} \\ = a_{21} & * & * \\ = a_{31} & * & * \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ a_{21} & * & * \\ a_{31} & * & * \end{bmatrix} \Rightarrow \begin{bmatrix} L_{11} & U_{12} & U_{13} \end{bmatrix}$$

$a_{11} : L_{11}$

$a_{21} : L_{21}$

$a_{31} : L_{31}$

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$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

Option - 2

$l_{11} = a_{11}$
$l_{21} = a_{21}$
$l_{31} = a_{31}$

$U_{12} = \frac{a_{12}}{a_{11}}$

$U_{13} = \frac{a_{13}}{a_{11}}$

Turn A → into an upper matrix.

① Consider a_{11} as a Pivot

Divide $\frac{a_{21}}{a_{11}}$ → get value

$\frac{a_{31}}{a_{11}}$ get value

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow \left(-\frac{a_{21}}{a_{11}} R_1 + R_2\right) \\ \rightarrow \left(-\frac{a_{31} R_1}{a_{11}} + R_3\right) \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \left(-\frac{a_{21}}{a_{11}} a_{12} + a_{22}\right) & \left(-\frac{a_{21}}{a_{11}} a_{13} + a_{23}\right) \\ 0 & \left(-\frac{a_{31}}{a_{11}} a_{12} + a_{32}\right. & \left(-\frac{a_{31}}{a_{11}} a_{13} + a_{33}\right. \end{bmatrix}$$

$$\begin{bmatrix} L_{11} \\ L_{21} \\ L_{31} \end{bmatrix}$$

$-\frac{L_{21}}{L_{11}} R_1 + R_2$

option - 2

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$L_{11} = a_{11}$
$L_{21} = a_{21}$
$L_{31} = a_{31}$

$U_{12} = \frac{a_{12}}{a_{11}}$
$U_{13} = \frac{a_{13}}{a_{11}}$

$-\frac{L_{31}}{L_{11}} R_1 + R_3$

$$\begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & -\frac{L_{21}}{L_{11}}(\cancel{L_{11}}U_{12}) + \cancel{L_{21}}U_{12} + L_{22} & -\frac{L_{21}}{L_{11}}(\cancel{L_{11}}U_{13}) + \cancel{L_{21}}U_{13} + L_{22}U_{23} \\ 0 & -\frac{L_{31}}{L_{11}}(\cancel{L_{11}}U_{12}) + \cancel{L_{31}}U_{12} + L_{32} & -\frac{L_{31}}{L_{11}}(\cancel{L_{11}}U_{13}) + \cancel{L_{31}}U_{13} + L_{32}U_{23} + L_{33}) \end{bmatrix}$$

ALL Equal values are deducted

$$\begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & L_{22} & L_{22}U_{23} \\ 0 & L_{32} & L_{32}U_{23} + L_{33} \end{bmatrix}$$

get L_{22}, L_{32}

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$$A_1 = \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & \boxed{L_{22}} & L_{22}U_{23} \\ 0 & L_{32} & l_{32}U_{23} + L_{33} \end{bmatrix} \begin{array}{l} \rightarrow \text{Same} \\ \rightarrow \text{Same} \\ \rightarrow -\frac{\ell_{32}}{L_{22}}R_2 + R_3 \end{array} \Bigg\} \Rightarrow U_1$$

Divide

Use L_{22} as a pivot divide $\frac{\ell_{32}}{L_{22}} \rightarrow$ multiply $(-1)R_2$

$$U_1 = \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & L_{22} & L_{22}U_{23} \\ 0 & 0 & -\frac{\ell_{32}}{L_{22}}(\cancel{L_{22}}\ U_{23}) + L_{32}\cancel{U_{23}} + L_{33} \end{bmatrix}$$

$$\Downarrow$$

$$U = \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & L_{22} & L_{22}U_{23} \\ 0 & 0 & L_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11} u_{12} & l_{11} u_{13} \\ 0 & \boxed{l_{22}} & l_{22} u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

divide

Divide

get L_{22}

Use l_{11} as a pivot $\Rightarrow$ Divide $\frac{l_{11} u_{12}}{l_{11}} = u_{12}$

l_{11} as a pivot $\rightarrow$ divide $\frac{l_{11} u_{13}}{l_{11}} = u_{13}$

l_{22} as a pivot $\rightarrow$ divide $\frac{l_{22} u_{23}}{l_{22}} = u_{23}$

Convert matrix A $\rightarrow$ upper matrix

Summary

For A_1 $\rightarrow$ get

$$\begin{bmatrix} l_{11} & l_{11} u_{12} & l_{11} u_{13} \\ 0 & l_{22} & l_{22} u_{23} \\ 0 & l_{23} & l_{32} u_{23} + l_{33} \end{bmatrix}$$

$\left.\begin{matrix} l_{22} \\ l_{23} \end{matrix}\right\}$ obtained

Convert $U_1 \to U_2$ get

$$U = \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ 0 & L_{22} & L_{22}U_{23} \\ 0 & 0 & L_{33} \end{bmatrix}$$

we have from previous U_1-matrix L_{11}, L_{22}

L_{32}

get L_{33}

U_{12}, U_{13}

U_{23}