Quadratic interpolation:

Quadratic interpolation is the process of using a 2nd-order polynomial to make an interpolation for a function.

This process provides an accuracy of the estimate which is better than the linear Interpolation, recall our general form of a polynomial

$P(x) = a_0 + a_1.x + a_2.x^2 + a_3.x^3 + \ldots + a_{n-1}.x^{n-1} + a_n.x^n$

for the quadratic form , we can rewrite the previous polynomial as

$P(x) = a_0 + a_{1.}x + a_2.x^2$

Suppose we have 3 points $A(x_0, y_0)$, $B(x_1, y_1)$, $C(x_2, y_2)$, and we want to fit a quadratic polynomial through these points. The general form of a quadratic polynomial is $P(x) = a_0 + a_1.x + a_2.x^2$. Thus, if we were to evaluate P(x) at these three points, we get three equations

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That can be written as follows

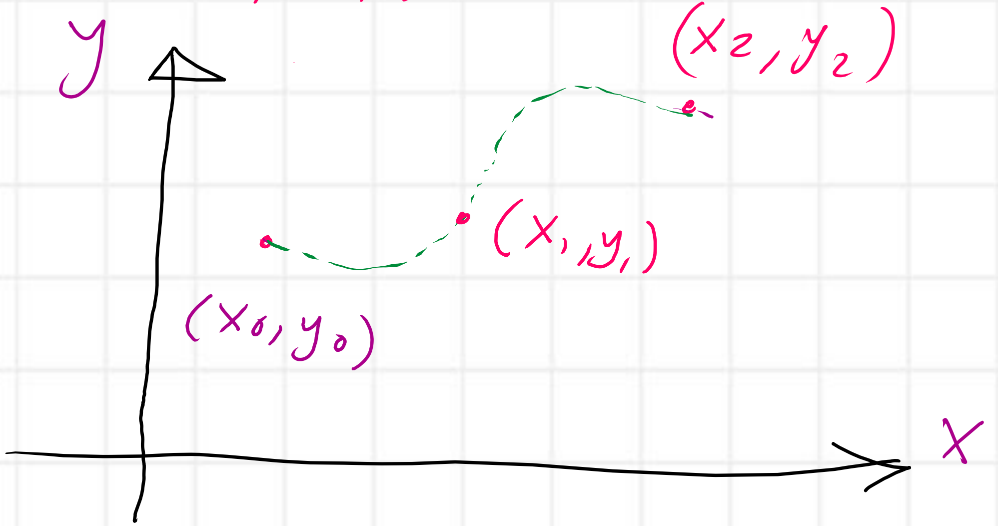
We need to estimate

a_0, a_1, a_2

$$P(x_0) = y_0 = a_0 + a_1 . x_0 + a_2 . x_0^2$$ For point A

$$P(x_1) = y_1 = a_0 + a_1 . x_1 + a_2 . x_1^2$$ For point B

$$P(x_2) = y_2 = a_0 + a_1 . x_2 + a_2 . x_2^2$$ For point C



Write these equations in matrix form

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

This matrix, is called Vandermonde matrix denoted by V

can be written in form of

$$\begin{array}{ccc} V_{1,1} & V_{1,2} & V_{1,3} \\ \end{array}$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

$V_{i,j}$ i : row

j : Columns

$V_{1,1} = (x_{i-1})^{j-1} = (x_{(1-1)})^{1-1} = x_0^0 = 1$

$V_{1,2} = (x_{(1-1)})^{2-1} = (x_{1-1})^1 = x_0^1 = x_0$

$i=1$

$j=2$

$V_{1,3} = (x_{1-1})^{3-1} = (x_{1-1})^2 = x_0^2 = x_0^2$

The first row items

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

$V_{2,1}$ $V_{2,2}$ $V_{2,3}$

$V_{i,j}$

i : row

j : Columns

$V_{2,1} = (x_{i-1})^{j-1} = (x_{(2-1)})^{1-1} = x_1^0 = 1$

$V_{2,2} = (x_{(2-1)})^{2-1} = (x_1)^1 = x_1^1 = x_1$

$i =$

$j = 2$

$V_{2,3} = (x_{2-1})^{3-1} = (x_1)^2 = x_1^2 = x_1^2$

The second row items

For the third row of Vandermonde : $V_{3,1} = x_2^0$, $V_{3,2} = x_2^1$ & $V_{3,3} = x_2^2$

To get the as value, we can write

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \frac{\text{Adjoin}}{\text{Determinant}} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

For the determinant, series of operations can be applied, as follows

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

To make $x_0 = 0$

A- multiply the 2nd column by x_0

B-Subtract 2nd column from the 3rd column.

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$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & x_0 & x_0^2 - x_0^2 \\ 1 & x_1 & x_1^2 - x_0 x_1 \\ 1 & x_2 & x_2^2 - x_0 x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & x_0 & 0 \\ 1 & x_1 & x_1^2 - x_0 x_1 \\ 1 & x_2 & x_2^2 - x_0 x_2 \end{bmatrix}$$

C- Multiply the 1st column by x_0

D- Subtract 1st column from the 2nd column.

$$\begin{bmatrix} 1 & x_0 - x_0 & 0 \\ 1 & x_1 - x_0 & (x_1)(x_1 - x_0) \\ 1 & x_2 - x_0 & x_2(x_2 - x_0) \end{bmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) \\ 1 & x_2 - x_0 & x_2(x_2 - x_0) \end{pmatrix}$$

to be $= 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) \\ 1 & x_2 - x_0 & x_2(x_2 - x_0) \end{pmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_2 \\ R_3 - R_1 \\ R_3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 - x_0 & x_1(x_1 - x_0) \\ 0 & x_2 - x_0 & x_2(x_2 - x_0) \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (x_1-x_0) & x_1(x_1-x_0) \\ 0 & (x_2-x_0) & x_2(x_2-x_0) \end{pmatrix}$$

The determinant

$$+1\,(x_1-x_0)(x_2-x_0)\,x_2-(x_1)(x_1-x_0)(x_2-x_0)$$

$$=(x_2-x_0)\left(x_2 x_1 - x_0 x_2 - x_1^2 + x_1 x_0\right)$$

$$D=(x_2-x_0)(x_1-x_0)(x_2-x_1)$$

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Adjoin matrix of Vandermonde:

Steps

A- Estimate the matrix of minors.

B-Turn the matrix of minors into cofactors.

3-Adjugate.

$$V=\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

$M_{11} = (x_1 x_2^2) - (x_2)(x_1^2) \rightarrow C_{11} = +M_{11}$

$M_{12} = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) \rightarrow C_{12} = -M_{12}$

$M_{13} = x_2 - x_1 = (x_2 - x_1) \rightarrow C_{13} = +M_{13}$

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Adjoin matrix of Vandermonde:

Steps

A- Estimate the matrix of minors.

B-Turn the matrix of minors to cofactors.

$M_{21} = x_0 x_2^2 - x_2 x_0^2 \rightarrow C_{21} = -M_{21}$

3-Adjugate.

$M_{22} = x_2^2 - x_0^2 = (x_2 - x_0)(x_2 + x_0) \rightarrow C_{22} = +M_{22}$

$M_{23} = (x_2 - x_0) \rightarrow C_{23} = -M_{23}$

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

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Adjoin matrix of Vandermonde:

Steps

A- Estimate the matrix of minors.

B-Turn the matrix of minors into cofactors.

3-Adjugate.

$$V=\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

$M_{31} = x_0 x_2^2 - x_2 (x_0)^2 \rightarrow C_{31} = M_{31}$

$M_{32} = x_1^2 - x_0^2 = (x_1 - x_0)(x_1 + x_0) \rightarrow C_{32} = -M_{32}$

$M_{33} = (x_1 - x_0) \rightarrow C_{33} = +M_{33}$

$$\begin{pmatrix} x_1 * x_2{}^2 - x_1{}^2 * x_2 & -(1 * x_2{}^2 - 1 * x_1{}^2) & 1 * x_2 - 1 * x_1 \\ -(x_0 * x_2{}^2 - x_0{}^2 * x_2) & 1 * x_2{}^2 - 1 * x_0{}^2 & -(1 * x_2 - 1 * x_0) \\ x_0 * x_1{}^2 - x_1 * x_0{}^2 & -(1 * x_1{}^2 - 1 * x_0{}^2) & 1 * x_1 - 1 * x_0 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Cofactor

$$\begin{pmatrix} x_1 x_2^2 - x_1^2 x_2 & x_2 (x_0^2 - x_2 x_0) & x_0 x_1^2 - x_1 x_0^2 \\ (x_1^2 - x_2^2) & x_2^2 - x_0^2 & -(x_1^2 - x_0^2) \\ x_2 - x_1) & -(x_2 - x_0) & x_1 - x_0 \end{pmatrix}$$

$(x_2 - x_1)$ Common factor

$\downarrow$ $(x_2 - x_0)$

$(x_1 - x_0)$ Common Factor

$$\begin{pmatrix} x_1 x_2^2 - x_1^2 x_2 & x_2 (x_0^2 - x_2 x_0) & x_0 x_1^2 - x_1 x_0^2 \\ (x_1^2 - x_2^2) & x_2^2 - x_0^2 & -(x_1^2 - x_0^2) \\ x_2 - x_1 & -(x_2 - x_0) & x_1 - x_0 \end{pmatrix}$$

factoring ↓

$$\begin{pmatrix} (x_2 - x_1) x_1 x_2 & (x_2 - x_0)(-)(x_0 x_2) & (x_1 - x_0)(x_1 x_0) \\ -(x_2 - x_1)(x_2 + x_1) & (x_2 - x_0)(x_2 + x_0) & -(x_1^2 - x_0^2) \\ (x_2 - x_1) & -(x_2 - x_0) & (x_1 - x_0) \end{pmatrix}$$

Our determinant $D = (x_2 - x_0)(x_1 - x_0)(x_2 - x_1)$

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$$Va = Y$$

For the inverse of Vandermonde.

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 \\ 1 & x_1 & {x_1}^2 \\ 1 & x_2 & {x_2}^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$$V^{-1} = \frac{1}{D} \left(\text{Cofactor} \right)$$

$$V^{-1} = \frac{\begin{pmatrix} (x_2 - x_1)x_1 x_2 & (x_2 - x_0)(-)(x_0 x_2) & (x_1 - x_0)(x_1 x_0) \\ -(x_2 - x_1)(x_2 + x_1) & (x_2 - x_0)(x_2 + x_0) & -(x_1^2 - x_0^2) \\ (x_2 - x_1) & -(x_2 - x_0) & (x_1 - x_0) \end{pmatrix}}{(x_2 - x_0)(x_1 - x_0)(x_2 - x_1)}$$

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$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

The process of finding the a's coefficients of the polynomial is not attractive; it involves the solution of algebraic equations.

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