Interpolating Polynomial by Matrices

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Linear algebra and its application

$\boxed{33}$

Chapter 1.2

Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data (1, 12), (2, 15), (3, 16). That is, find a_0, a_1, and a_2 such that

$$a_0 + a_1(1) + a_2(1)^2 = 12$$
$$a_0 + a_1(2) + a_2(2)^2 = 15$$
$$a_0 + a_1(3) + a_2(3)^2 = 16$$

Solution

The above equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 16 \end{bmatrix}$$

augmented matrix

$$\begin{array}{cccc} a_0 & a_1 & a_2 & \\ \left[\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{matrix} 1 \\ 4 \\ 9 \end{matrix} & \left|\begin{matrix} 12 \\ 15 \\ 16 \end{matrix}\right] \end{array}$$

Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data (1, 12), (2, 15), (3, 16). That is, find a_0, a_1, and a_2 such that

$$a_0 + a_1(1) + a_2(1)^2 = 12$$
$$a_0 + a_1(2) + a_2(2)^2 = 15$$
$$a_0 + a_1(3) + a_2(3)^2 = 16$$

$$\begin{array}{c} \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array}\right] \end{array} \longleftarrow \text{This is the augmented matrix}$$

Pivot

row.

$R_1 \to R_1$

$R_3 \to [-R_1 + R_3]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 1 & 3 & 9 & 16 \end{array}\right] \begin{array}{l} \to R_1 \\ \Rightarrow R_2 \\ \Rightarrow [-R_1 + R_3] \to R_3 \end{array}$$

Use as a pivot

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array}\right] \begin{array}{l} \to R_1 \\ \to R_2 \\ \to \underbrace{(-2R_2 + R_3)}_{R_3} \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

This a $\boxed{\text{REF}}$ $\Rightarrow$ Back substitute

$$\begin{array}{ccc|c} a_0 & a_1 & a_2 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$2a_2 = -2 \Rightarrow a_2 = -1$

$a_1 + 3a_2 = 3 \Rightarrow a_1 = 3 - 3(-1) = 6$

$a_0 + a_1 + a_2 = 12 \Rightarrow a_0 = 12 - 6 - (-1) = 7$

We could proceed to RREF – Gauss-Jordan

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right] \begin{array}{l} \to R_1 \\ \to R_2 \\ \to \frac{R_3}{2} \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \to (-R_3 + R_1) \to R_1 \\ \to (-3R_3 + R_2) \to R_2 \\ \to R_3 \end{array}$$

$$
\begin{array}{cccc}
a_0 & a_1 & a_2 & \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 12 \\
0 & 1 & 3 & 3 \\
0 & 0 & 1 & -1
\end{array}\right]
\begin{array}{l}
\to (-R_3 + R_1) \to R_1 \\
\to (-3R_3 + R_2) \to R_2 \\
\to R_3
\end{array}
\Rightarrow
\begin{array}{cccc}
a_0 & a_1 & a_2 & \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 1 & 0 & 13 \\
0 & 1 & 0 & 6 \\
0 & 0 & 1 & -1
\end{array}\right]
$$

pivot

Use as a pivot

$$
\begin{array}{l}
\to (-R_2 + R_1) \to R_1 \\
\to R_2 \\
\to R_3
\end{array}
\qquad
\begin{array}{cccc}
a_0 & a_1 & a_2 & \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & 6 \\
0 & 0 & 1 & -1
\end{array}\right]
$$

$$
\left.\begin{array}{l}
a_0 = 7 \\
a_1 = 6 \\
a_2 = -1
\end{array}\right\} \text{Same as } \boxed{\text{REF}}
$$

We have

$p(t) = a_0 + a_1 t + a_2 t^2$

$a_0 = 7, \; a_1 = 6, \; a_2 = -1$

Check using $p(t)$

$t, \; p(t)$

check $(1, 12)$ point

$p(t) = 7 + 6t - t^2$

$p(1) = 7 + 6(1) - (1)^2 = 13 - 1 = 12$ ok

$$\begin{array}{cc} t & f(t) \end{array}$$

* Check point $(2, 15) \rightarrow p(2) = 7 + 6(2) - (2)^2$

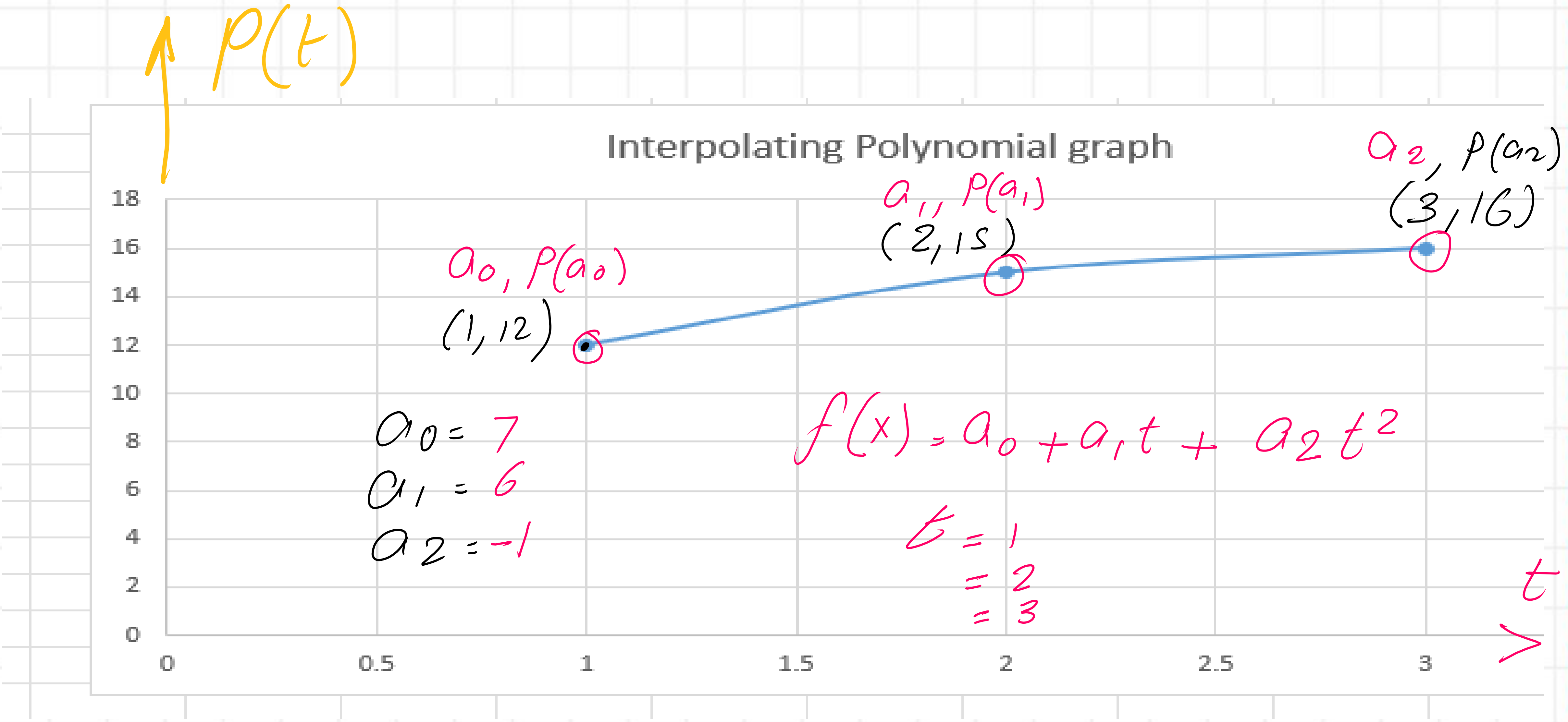
$= 7 + 12 - 4 = \boxed{15}$ ok $= f(2)$

* Check point $(3, 16) \rightarrow p(3)$

$$\begin{array}{cc} t & f(t) \end{array}$$

$= 7 + 6(3) - (3)^2$

$= 7 + 18 - 9 = 16$ ok $= f(3)$



Graph between t & P(t)

31. (***Calculus Required***) Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$, where $f(x) = e^{2x}$.

Solution source : Elementary Linear algebra with application ⇒ Bernard Kolman

Section : 2.2

$f(x) = e^{2x} \Rightarrow f'(x) = 2e^{2x} \Rightarrow p'(x)$ at x

$= p(x)_{x=0}$ $\quad f''(x) = 4e^{2x} \rightarrow p''(x)$ at

Polynomial $p(x) = ax^2 + bx + c$

Three unkowns

a, b, c

31. (*Calculus Required*) Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$, where $f(x) = e^{2x}$.

Solution

$$p(x) = ax^2 + bx + c = e^{2x} \quad (1)$$

$$p'(x) = 2ax + b + 0 = 2e^{2x} \quad (2)$$

$$p''(x) = 2a + 0 + 0 = 4e^{2x} \quad (3)$$

For three Conditions

$\Downarrow$

$$f(x) = e^{2x}$$

$$p(0) = f(0)$$

$$p'(0) = f'(0)$$

$$p''(0) = f''(0)$$

$$P(x) = ax^2 + bx + c = e^{2x} \quad (1)$$

$$P'(x) = 2ax + b + 0 = 2e^{2x} \quad (2)$$

$$P''(x) = 2a + 0 + 0 = 4e^{2x} \quad (3)$$

For $x = 0$

$$a(0) + b(0) + c = 1$$

$$a(0) + b(1) + 0c = 2$$

$$a(2) + b(0) + 0c = 4$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

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$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 4 \end{array}\right] \end{array} \quad R_1 \leftrightarrow R_3$$

swap $R_3 \to R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \left(\frac{1}{2}R_1\right) \to R_1 \Rightarrow R_2 \to R_3 \quad \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

RREF

From R_3 $(0)a + (0)b + c = 1 \Rightarrow c = 1$

R_2 $(0)a + (1)b + (0)c = 2$ $\quad b = 2$

R_1 $1a + (0)b + 0(c) = 2$ $\quad a = 2$

$$P(x) = ax^2 + bx + c \Rightarrow \begin{Bmatrix} 2x^2 + 2x + 1 \\ 4x + 2 \\ 4 \end{Bmatrix}$$

$p'(x) \Rightarrow$

$f''(x) \to$

orginal

$$\begin{bmatrix} e^{2x} \\ 2e^{2x} \\ 4e^{2x} \end{bmatrix}$$

$\Rightarrow$ Final Expresions

3rd practice

32 **(Calculus Required)** Construct a linear system of equations to determine a quadratic polynomial $p(x) = ax^2 + bx + c$ that satisfies the conditions $p(1) = f(1)$, $p'(1) = f'(1)$, and $p''(1) = f''(1)$, where $f(x) = xe^{x-1}$.

Solution : Solve for a, b, c values based on given conditions

$$P(x) = ax^2 + bx + c$$

$$P'(x) = 2ax + b$$

$$P''(x) = 2a$$

$$\rightarrow f(x) = x e^{x-1}$$

$$f'(x) = x \frac{d(e^{x-1})}{dx} + e^{(x-1)}(1)$$

$$f'(x) = (x+1) e^{x-1}$$

$$f''(x) = (x+1)e^{x-1} + e^{x-1}(1)$$

$$= e^{x-1}[x+2]$$

Source: Bernard Kolman

Elementary Linear Algebra with applications Ninth's edition

Sec 2.2

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for $x = 1$ we have

$P(x) = ax^2 + bx + c$

$P(1) = a(1)^2 + b(1) + c = f(1) = e^0 = 1$ I

$P'(1) = 2a(1) + b = f'(1) = 2e^0 = 2$ II

$P''(1) = 2a = f''(1) = 3e^0 = 3$ III

3ree Equations

$a + b + c = 1$

$2a + b + 0c = 2$

$2a + 0b + 0c = 3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{array}{cccc} a & b & c & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 3 \end{array}\right]$$

augmented form

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$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 0 & 0 & 3 \end{array}\right] \end{array}$$

$R_1 \Rightarrow R_3$

swap $R_3 \rightarrow R_1$

$$\Rightarrow R_2 \quad \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

pivot

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

0

0

$\rightarrow (R_1) \rightarrow$

$\rightarrow (-1 R_2 + R_1) \rightarrow R_2$

$(-2 R_3 + R_1) \rightarrow R_3$

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & -1 & 0 & +1 \\ 0 & -2 & -2 & 1 \end{array}\right] \end{array} \rightarrow -R_2$$

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & -2 & 1 \end{array}\right] \end{array}$$

$\rightarrow R_1$

$\rightarrow R_2$

$\rightarrow +\frac{1}{2} R_3 + R_1 \xrightarrow{} R_3$

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1/2 \end{array}\right] \end{array} \rightarrow \frac{-R_3}{R_3}$$

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$$
\begin{array}{c}
\begin{array}{ccc} a & b & c \end{array} \\
\left[\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & \tfrac{1}{2} \end{array}\right]
\begin{array}{l} \rightarrow R_1/2 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{ccc} a & b & c \end{array} \\
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right]
\end{array}
$$

$\boxed{\text{RREF}}$

Final values

$$
\left.\begin{array}{l} c = 1/2 \\ b = -1 \\ a = 3/2 \end{array}\right\}
$$

check $\left[\frac{3}{2} + (-1) + \frac{1}{2}\right) = 1$ ok

$2(3/2) + (-1) = 2$ ok

$2\left(\frac{3}{2}\right) = 3$ ok

$$P(X) = \frac{3}{2}X^2 - x + 1/2 \Rightarrow$$

$p'(x) = 3X - 1$

$p''(x) = 3X$

Final Expresion

$$
\left[\begin{array}{l} xe^{x-1} \\ (x+1)e^{x-1} \\ (x+2)e^{x-1} \end{array}\right.
$$