Introduction to LU Decomposition

Using Crout's method (2X2)

(a) Assumptions $U_{11} = U_{22} = 1$

(b) Convert $A \rightarrow$ Lower matrix option-1 $\rightarrow \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$

(c) Convert $A \rightarrow$ upper matrix option-2 $\begin{bmatrix} l_{11} & L_{11}U_{12} \\ 0 & L_{22} \end{bmatrix}$

(d) Use Elementary matrices

Solve two simultaneous Equations For x, y

Lu Decomposition   **Crout's method**

It is possible to show that any square matrix **A** can be expressed as a product of a lower triangular matrix *L* and an upper triangular matrix *U*.

$$A = L \cdot U$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

We have six unkowns $\rightarrow l_{11}$ $\quad U_{11}$

l_{21} $\quad U_{12}$

l_{22} $\quad U_{22}$

Crout's method $\rightarrow U_{11} = 1$

$U_{22} = 1$

$\Downarrow$ We have

l_{11} & l_{21} & l_{22}, U_{12} ?

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Crout's method LU Factorization

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$L_{11} + 0 = a_{11}$ $\boxed{1}$ $\rightarrow L_{11} = a_{11}$

$L_{21} + L_{22}(0) = a_{21}$ $\boxed{2}$ $\quad L_{21} = a_{21}$

$L_{11} U_{12} + 0 = a_{12}$ $\boxed{3}$ From 3 $a_{11}(U_{12}) = a_{12} \Rightarrow$

$U_{12} = a_{12}/a_{11}$

$= a_{12}/L_{11}$

$L_{21} U_{12} + L_{22} = a_{22}$ $\boxed{4}$ 4

$a_{21}\left(\frac{a_{12}}{a_{11}}\right) + L_{22} = a_{22} \Rightarrow L_{22} = a_{22} - \left(\frac{a_{12}}{a_{11}}\right) a_{21}$

divide

Keep

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow U = \begin{array}{cc} U_{11} & U_{12} \\ \left[\begin{matrix} 1 & \frac{a_{12}}{a_{11}} \\ 0 & 1 \end{matrix}\right] & \\ & U_{22} \end{array}$$

$$l = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} - U_{12}(L_{21}) \end{bmatrix}$$

How Can we get these value thru Elementary matrices and row operations?

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option_1 Convert $A \rightarrow$ lower matrix

$$\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12} + l_{22} \end{bmatrix}$$

$L \qquad U$

$\Rightarrow$ zero

$$\left[-\frac{c_{12}}{c_{11}} c_{11} + c_{12} \Rightarrow 0\right.$$

$$\left(-\frac{a_{21}}{a_{11}} \cdot a_{12} + a_{21}\right)$$

Crout's method

$$\begin{bmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12} + l_{22} \end{bmatrix} \begin{bmatrix} 1 & -\frac{l_{11}u_{12}}{l_{11}} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$$

$\Downarrow$

L Matrix

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Option -1 Convert A → to Lower matrix L

$$\begin{matrix} 0 & 0 \\ L & \end{matrix}$$

A (3 → Zero)

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$\frac{-\frac{3}{2}C_1 + C_2}{C_2}$$

$$\begin{bmatrix} 2 & -\frac{3}{2}(2)+3 \\ 3 & -\frac{3}{2}(3)+4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & -\frac{1}{2} \end{bmatrix}$$

$$L_{11} = 2$$
$$L_{21} = 3$$
$$L_{22} = -\frac{1}{2}$$

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$$

option -1 U?

Elementary matrix

$$L = A \cdot E_1$$

right side of matrix

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}}_{E_1} = \underbrace{\begin{bmatrix} 2 & 0 \\ 3 & -\frac{1}{2} \end{bmatrix}}_{L}$$

How can we get U?

$$U = \begin{bmatrix} 1 & +3/2 \\ 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & +3/2 \\ 0 & 1 \end{bmatrix} = U$$

$$A \; E_1 \rightarrow L$$

$$L = (L \; U) E_1$$

$$L^{-1} \cdot L = L^{-1} \cdot L \; U \; E_1 \; / E_1$$

$$E_1^{-1} \; I = U \; (E_1 \; E_1^{-1})$$

$$E_1^{-1} = U \; I$$

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Option-2 Convert A → into upper matrix

$$\overbrace{\begin{bmatrix} L_{11} & L_{11}U_{12} \\ L_{21} & L_{21}U_{12}+L_{22} \end{bmatrix}}^{} \Rightarrow \begin{bmatrix} 1 & 0 \\ -\frac{L_{21}}{L_{11}} & 1 \end{bmatrix} \overbrace{\begin{bmatrix} L_{11} & L_{11}U_{12} \\ L_{21} & L_{21}U_{12}+L_{22} \end{bmatrix}}^{A}$$

Zero

$$\begin{bmatrix} L_{11} & L_{11}U_{12} \\ 0 & -L_{21}U_{12}+L_{21}U_{12}+L_{22} \end{bmatrix} \Rightarrow \quad -\frac{C_{21}}{C_{11}}C_{11}+C_{21}$$

$$\begin{bmatrix} L_{11} & L_{11}U_{12} \\ 0 & L_{22} \end{bmatrix}$$

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Option - 2 Convert $A \rightarrow$ to an upper matrix

A

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ -\frac{3}{2}R_1 + R_2 \\ \overline{\quad R_2 \quad} \end{array} \begin{bmatrix} 2 & 3 \\ -\frac{3}{2}(2)+3 & -\frac{3}{2}(3)+4 \end{bmatrix}$$

Zero

$$\begin{bmatrix} 2 & 3 \\ 0 & -1/2 \end{bmatrix} \Rightarrow \begin{array}{l} L_{11} = 2 \\ L_{11} U_{12} = 3 \Rightarrow U_{12} = \frac{3}{2} \\ L_{22} \end{array}$$

option-2 $\Rightarrow A \rightarrow$ upper matrix

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

L_{11} A_1

$L_{11}(U_{12})$

L_{22}

$L_{11} = 2$

$L_{22} = -1/2$

$L_{11} U_{12} = 3$

$U_{12} = +\frac{3}{2} = 3/2$

For the two sets of equations : 2x+3y=13 & 3x+4y=18

Find x, y

Crout's method

$$A \quad X = b$$

$$\overset{A}{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{b}{\begin{bmatrix} 13 \\ 18 \end{bmatrix}}$$

$$\underset{L}{\begin{bmatrix} 2 & 0 \\ 3 & -1/4 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & +\frac{3}{2} \\ 0 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

Take $Ux = C$

Rewrite $LC = b$

L^{-1}

$$L^{-1} = \begin{bmatrix} 2 & 0 \\ 3 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ -3 & 2 \end{bmatrix} \Big/ -1 = \begin{bmatrix} +\frac{1}{2} & 0 \\ 3 & -2 \end{bmatrix}$$

$$|L| = -1$$

$$L^{-1} L C = L^{-1} b \Rightarrow C = \begin{bmatrix} \frac{1}{2} & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{13}{2} \\ 3 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 1$

$$U = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

$$UX = C \qquad U^{-1}.U.X = U^{-1}.C$$

$$I.X = U^{-1}\, C$$

$$\begin{array}{c} \\ \begin{bmatrix} x \\ y \end{bmatrix} \end{array} = \begin{array}{c} \\ \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} \frac{13}{2} \\ 3 \end{bmatrix} \end{array} = \begin{array}{c} \\ \begin{bmatrix} \frac{13}{2} - \frac{9}{2} \\ 3 \end{bmatrix} \end{array} = \begin{array}{c} X \\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{array}$$

Check:

$$2x + 3y = 13 \rightarrow (2)(2) + 3(3) = 13 \quad \text{R.H.S}$$

$$3x + 4y = 18 \qquad 3(2) + 4(3) = 18$$

R.H.S

We could use $LU = A$

We have $Ax = b$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

$$L \cdot U \cdot x = b$$

$$U^{-1} \underbrace{L^{-1} L} U \cdot x = U^{-1} L^{-1} b$$

$$I \cdot x = U^{-1} L^{-1} b$$

$$X = \underbrace{\overset{E_1}{\begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix}}}_{U^{-1}} \underbrace{\begin{bmatrix} 1/2 & 0 \\ 3 & -2 \end{bmatrix}}_{L^{-1}} \underbrace{\begin{bmatrix} 13 \\ 18 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & +3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix} = \begin{bmatrix} -52+54 \\ 39-36 \end{bmatrix} = \overset{x}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\left.\begin{array}{l} x = 2 \\ y = 3 \end{array}\right\} \text{ Same result}$$

If we have for two set of equations : 2x+3y=13

3x+4y=18

$$\left.\begin{aligned} 2x+3y&=13\\ 3x+4y&=18 \end{aligned}\right\} \quad x=2,\; y=3$$

Our Final Check :

$$2(2)+3(3)=4+9=13 \rightarrow ok$$

$$3(2)+4(3)=6+12=18 \rightarrow ok$$

if $X = U^{-1} L^{-1} (b) \Rightarrow$ then: $\begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ Same result for x, y by using vector C

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