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25–28 ■ Back-Substitution A matrix is given in row-echelon form. **(a)** Write the system of equations for which the given matrix is the augmented matrix. **(b)** Use back-substitution to solve the system.

25. $\begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 1 & -3 & 8 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 2 & 3 & -1 & 7 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & -2 & 2 & 5 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Two Examples are Considered

25–28 ■ Back-Substitution A matrix is given in row-echelon form. **(a)** Write the system of equations for which the given matrix is the augmented matrix. **(b)** Use back-substitution to solve the system.

25. $\begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ → Solution: write in the augmented form

$$\begin{array}{ccccc} x & y & z & & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow$$

Part (a)

Simultaneous Equations

$Ax = b$

$$\begin{aligned} x - 2y + 4z &= 3 \\ 0x + y + 2z &= 7 \\ 0x + 0y + z &= 2 \end{aligned}$$

Now: These equation are in a triangular form

$$\begin{array}{cccccc} & x & y & z & & \\ \textbf{25.} & \left[\begin{array}{ccc|c} 1 & -2 & 4 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] & & & & \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \end{array}$$

Part (b) Use back substitution

$Z = 2 \rightarrow$ Back to II

$0x + y + 2Z = 7 \rightarrow y = 7 - 2Z$

$y = 7 - 2(2) = 3$

Back to (I) we have $x - 2y + 4Z = 3$

$Z = 2$
$y = 3$ $\rightarrow x = 3 + 2y - 4Z$

$= 3 + (2)(3) - 4(2) = 3 + 6 - 8$

$x = 1$

R.H.S

Check in any Equation: $x - 2y + 4Z = 1 - 6 + 8 = 3$

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27. $\begin{bmatrix} 1 & 2 & 3 & -1 & 7 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ $\rightarrow$ a

augmented form

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 7 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

These are the Equations

$$\left.\begin{aligned} x + 2y + 3z - w &= 7 \quad (1) \\ 0x + y - 2z + 0w &= 5 \quad (2) \\ 0x + 0y + z + 2w &= 5 \quad (3) \\ 0x + 0y + 0z + w &= 3 \quad (4) \end{aligned}\right\}$$

from 4

$w = 3$

Back substitute in 3

$$\begin{array}{ccccccc} & x & y & z & w & & \\ \textbf{27.} & \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 3 \\ -2 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 2 \\ 1 \end{matrix} & \begin{matrix} | \\ | \\ | \\ | \end{matrix} & \left.\begin{matrix} 7 \\ 5 \\ 5 \\ 3 \end{matrix}\right] \end{array}$$

$$\left.\begin{aligned} x + 2y + 3z - w &= 7 \\ 0x + y - 2z + 0w &= 5 \\ 0x + 0y + z + 2w &= 5 \\ 0x + 0y + 0z + w &= 3 \end{aligned}\right\} \quad \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

We have

$w = 3$

From (3)

$z + 2w = 5$

$z = 5 - 2w = 5 - 2(3)$

$z = -1$

Back substitute in (2)

$w = 3$

$z = -1$

$y - 2z + 0w = 5$

$y = 5 + 2z \rightarrow y = 5 + 2(-1) = +3 \nearrow y = 3$

Back substitute in (1)

Finally: $x + 2y + 3z - w = 7 \Rightarrow x = -2y - 3z + w + 7$

$x = -2(3) - 3(-1) + 3 + 7 = -6 + 3 + 3 + 7$

$x = +7$

Check the Four Equations

27 b

$$\left.\begin{aligned} x + 2y + 3z - w &= 7 \\ 0x + y - 2z + 0w &= 5 \\ 0x + 0y + z + 2w &= 5 \\ 0x + 0y + 0z + w &= 3 \end{aligned}\right\}\quad \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Using

$x = 7$

$y = 3$

$z = -1$

$w = 3$

$\boxed{1}$ $7 + 2(3) + 3(-1) - 3 = 7 + 6 - 3 - 3 = +7$ R.H.S

$\boxed{2}$ $0(7) + 1(3) - 2(-1) + 0(3) = 0 + 3 + 2 + 0 = 5 \rightarrow$ R.H.S

$\boxed{3}$ $0(7) + 0(3) + 1(-1) + 2(3) = 0 + 0 - 1 + 6 = 5 \rightarrow$ R.H.S

$\boxed{4}$ $0(7) + 0(3) + 0(-1) + 1(3) = 0 + 0 + 0 + 3 = 3$

$\downarrow$

R.H.S