A matrix is said to be in **row echelon form** Sec 1.2 Page 29

(i) If the first nonzero entry in each nonzero row is 1.

(ii) If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.

(iii) If there are rows whose entries are all zero, they are below the rows having nonzero entries.

i start with 1 as first non zero Entry row

All zeros must be placed below rows of nonzeros
iii

DAvid C. Lay

REF From 1-3

RREF From 1-3 & 4-5

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

DAVID C. LAY

Definition: A matrix is in row echelon form (REF) if it satisfies the following three properties:

→ states non zero

(i) All nonzero rows are above any rows of all zeros.

Mnemonic: "Rows of zeros have to be at the bottom!"

(ii) Each leading (nonzero) entry of a row is in a column to the right of the leading (nonzero) entry of the row above it.

Mnemonic: "As you go down, leading entries must move to the right!"

(iii) All entries in a column below a leading (nonzero) entry are zeros.

Mnemonic: "Entries below leading entries must be zero!"

Definition: A matrix is in reduced row echelon form (RREF) if it satisfies the following three properties:

(i)–(iii) It is in REF.

(iv) The leading (nonzero) entry in each row is 1. ← must Equal 1

Mnemonic: "Leading entries must equal 1!"

(v) Each leading 1 is the only nonzero entry in its column.

Mnemonic: "Entries above leading entries must also be zero!"

Case-1 as r.E.F

First Leading pivot is non Zero, First Entry is 1

$$\begin{array}{ll} & \begin{matrix} 1 & a & b \end{matrix} \\ \text{row } k \quad 0\text{'s} = 1 & \begin{matrix} 0 & 1 & d \end{matrix} \\ \text{row } k+1 \quad 0\text{'s} = 2 & \begin{matrix} 0 & 0 & 1 \end{matrix} \end{array}$$

Row-echelon form

2nd Leading

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & \boxed{0} & 1 \end{bmatrix}$$

Case-1

Row-echelon form

First Leading

$$\begin{bmatrix} 1 & a & b \\ \boxed{0} & 1 & d \\ \boxed{0} & \boxed{0} & 1 \end{bmatrix}$$

Zeros are below the pivot

Zero

2nd Leading : First non-Zero Element in-Row-2

→ Located below and to the right from-1st-Leading

→ Colum 2, one Zero below the 2nd Leading

$$\begin{matrix} 1 & & \\ 0 & 1 & \\ & 0 & \end{matrix}$$

First non zero Element 2nd row

This is how the process might work for a 3×4 matrix:

▨ has a value

$$\begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & \blacksquare & \blacksquare & \blacksquare \\ 0 & 1 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare & \blacksquare \end{bmatrix} \quad \begin{bmatrix} 1 & \boxed{\blacksquare} & \blacksquare & \blacksquare \\ 0 & 1 & \boxed{\blacksquare} & \blacksquare \\ 0 & 0 & 1 & \boxed{\blacksquare} \end{bmatrix}$$

Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution. This technique is called **Gaussian elimination**, in honor of its inventor, the German mathematician C. F. Gauss (see page 326).

What is a pivot column?

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

$$\left[\begin{array}{ccc|c} 1 & ▨ & ▨ & ▨ \\ 0 & 1 & ▨ & ▨ \\ 0 & 0 & 1 & ▨ \end{array}\right]$$

First, 2nd, 3rd pivot Columns

RREF

Reduced Row Echelon Method ⇒ Gauss - Jordan

Another matrix method for solving systems is the **reduced row echelon method.** Earlier, we saw that the row echelon form of a matrix has 1s along the main diagonal and 0s below. ***The reduced row echelon form has 1s along the main diagonal and 0s both below and above.*** For example, the augmented matrix of the system

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 5 \\ 3x + y - 2z &= 9 \end{aligned} \quad \text{is} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 5 \\ 3 & 1 & -2 & 9 \end{array}\right].$$

By using row transformations, this augmented matrix can be transformed to

Reduced row echelon form

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right], \quad \text{which represents the system} \quad \begin{aligned} x &= 3 \\ y &= 2 \\ z &= 1. \end{aligned}$$

Zeros are above and below 1's

The solution set is $\{(3, 2, 1)\}$. There is no need for back-substitution with reduced row echelon form.

Three pivots = Three pivot Colums

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

DAVID C. LAY

EXAMPLE 2 Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \rightarrow REF$$

Definition: A matrix is in row echelon form (REF) if it satisfies the following three properties:

(i) All nonzero rows are above any rows of all zeros.

Mnemonic: "Rows of zeros have to be at the bottom!"

(ii) Each leading (nonzero) entry of a row is in a column to the right of the leading (nonzero) entry of the row above it.

Mnemonic: "As you go down, leading entries must move to the right!"

(iii) All entries in a column below a leading (nonzero) entry are zeros.

Mnemonic: "Entries below leading entries must be zero!"

Solution

Zero Can not be a pivot

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$R_4 \rightarrow R_1$

$$\approx \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

1 is the First pivot $\rightarrow$ $(-1, -2, 1)$ to be Zeros

$R_2 + \left(\frac{+1}{1}\right) R_1 \rightarrow R_2$, $R_3 + \left(\frac{+2}{1}\right) R_1 \rightarrow R_3$

$R_4 + (1) R_1 \rightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \begin{matrix} \rightarrow \text{Same} \\ \rightarrow R_2 + R_1 \rightarrow R_2 \\ \rightarrow R_3 + 2R_1 \rightarrow R_3 \\ \rightarrow \end{matrix} \Rightarrow \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & +2 & 4 & -6 & -6 \\ 0 & +5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

2nd pivot → R_2 & C_2 is non zero ok

↳ below to be zeros $R_3 - \frac{5}{2} R_2 \rightarrow R_3$

$R_4 + \frac{3}{2} R_2 \rightarrow R_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \quad R_4 \rightarrow R_3$$

because zeros should be at the bottom

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Columns are C_1 & C_2 & C_4

Pivots are 1 & 2 & -5

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

original matrix

$$A = \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & & c_4 & \end{array} \\ \begin{bmatrix} \boxed{0} & -3 & -6 & 4 & 9 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ -2 & -3 & 0 & \boxed{3} & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \end{array} \tag{3}$$

Pivot columns ■