Quadratic interpolation:

Quadratic interpolation is the process of using of 2nd order polynomial to make interpolation for a function.

This process provides an accuracy of the estimate which is better than the linear Interpolation, recall our general form of a polynomial

$$P(x) = a_0 + a_1.x + a_2.x^2 + a_3.x^3 + \dots + a_{n-1}.x^{n-1} + a_n.x^n$$

for the quadratic form , we can rewrite the previous polynomial as

$$P(x) = a_0 + a_1.x + a_2.x^2$$

Suppose we have 3 points A(x_0, y_0), B (x_1,y_1), C (x_2, y_2) and we want to fit a quadratic polynomial through these points, The general form of a quadratic polynomial is P(x) =$a_0+a_1.x+a_2.x^2$. Thus, if we were to evaluate P(x) at these three points, we get three equations

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That can be written as follows

$P(x_0) = y_0 = a_0 + a_1 .\, x_0 + a_2 .\, \overline{x_0}^{\,2}$ For point A

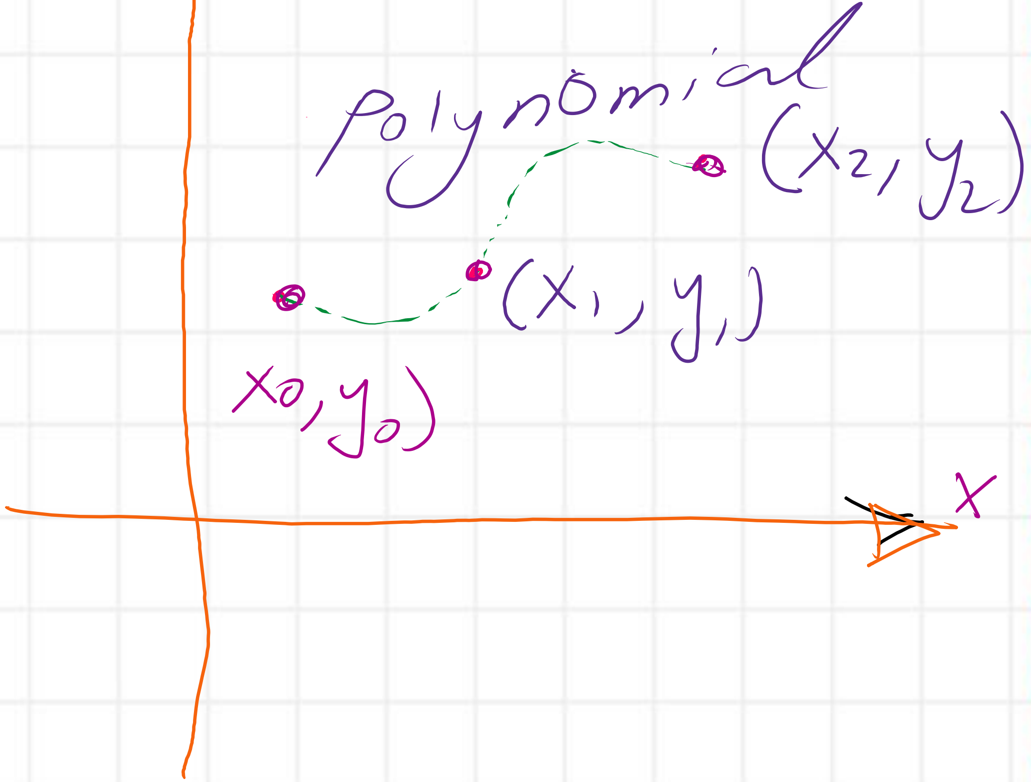
$P(x_1) = y_1 = a_0 + a_1 .\, x_1 + a_2 .\, \overline{x_1}^{\,2}$ For point B

$P(x_2) = y_2 = a_0 + a_1 .\, x_2 + a_2 .\, \overline{x_2}^{\,2}$ For point C

Write these equations in the matrix form

$$\begin{bmatrix} 1 & x_0 & x_0{}^\wedge 2 \\ 1 & x_1 & x_1{}^\wedge 2 \\ 1 & x_2 & x_2{}^\wedge 2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

we need to find a_0, a_1, a_2



This is Vandermonde matrix V

can be written in form of

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

The process of finding the a's coefficients of the polynomial is not attractive, it involves the solution of algebraic equations.

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Solved example-1

Find the value of f(x), when x=2.70.

Derive an expression of a polynomial for the following data,

X	1	2	3
f(x)	3	5	8

x_0, y_0 $\quad x_1, y_1$ $\quad x_2, y_2$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ \end{array}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

Solution

From the given table we have

$x_0 = 1$ & $y_0 = 3$

$x_1 = 2$ & $y_1 = 5$

$x_2 = 3$ & $y_2 = 8$

For First row b_{11}, b_{12}, b_{13} Elements

b_{11}

$$\left(\frac{x_1 x_2}{(x_2-x_0)(x_1-x_0)}\right) = \frac{2(3)}{(3-1)(2-1)} = \frac{6}{2(1)} = 3$$

$$b_{12} = \frac{-x_0 x_2}{(x_1-x_0)(x_2-x_1)} = \frac{-1(3)}{(2-1)(3-2)} = \frac{-3}{1(1)} = -3$$

$$b_{13} = \frac{x_0(x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1(2)}{2(3-2)} = \frac{2}{2} = +1$$

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Solved example-1

Find the value of f(x), when x=2.70.

Derive an expression of a polynomial for the following data,

X	1	2	3
f(x)	3	5	8

2nd row

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

Solution

From the given table we have

$x_0 = 1$ & $y_0 = 3$

$x_1 = 2$ & $y_1 = 5$

$x_2 = 3$ & $y_2 = 8$

<For the second row> b_{21}, b_{22}, b_{23}

$b_{21} \rightarrow \frac{-(x_2+x_1)}{(x_2-x_0)(x_1-x_0)} = \frac{-(3+2)}{(3-1)(2-1)} = \frac{-5}{2}$

$b_{22} = \frac{(x_2+x_0)}{(x_1-x_0)(x_2-x_1)} = \frac{(3+1)}{(2-1)(3-2)} = 4$

$b_{23} = \frac{-(x_1+x_0)}{(x_2-x_0)(x_2-x_1)} = \frac{-(1+2)}{(3-1)(3-2)} = \frac{-3}{2}$

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Solved example-1 **Find the value of f(x), when x=2.70.**

Derive an expression of a polynomial for the following data,

X	1	2	3
f(x)	3	5	8

3rd row

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

Solution

From the given table we have

$x_0 = 1$ & $y_0 = 3$

$x_1 = 2$ & $y_1 = 5$

$x_2 = 3$ & $y_2 = 8$

<For the third row> b_{31}, b_{32}, b_{33}

$b_{31} \rightarrow \frac{1}{(x_2-x_0)(x_1-x_0)} = \frac{1}{(3-1)(2-1)} = \frac{1}{2}$

$b_{32} = \frac{-1}{(x_1-x_0)(x_2-x_1)} = \frac{-1}{(2-1)(3-2)} = \frac{-1}{1}$

$b_{33} = \frac{1}{(x_2-x_0)(x_2-x_1)} = \frac{1}{(3-1)(3-2)} = \frac{1}{2}$

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Find the value of f(x), when x=2.70.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{x_1x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix}}_{b} * \underbrace{\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}}_{y}$$

	x_0	x_1	x_2
X	1	2	3
f(x)	3	5	8
	y_0	y_1	y_2

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} = \begin{bmatrix} 3(3) - 3(5) + 1(8) \\ -\frac{5}{2}(3) + 4(5) - \frac{3}{2}(8) \\ \frac{1}{2}(3) - 1(5) + \frac{1}{2}(8) \end{bmatrix} \begin{matrix} = +2 \\ = \frac{40-39}{2} \\ = \frac{3-10+8}{2} \end{matrix}$$

(3×3) (3×1) $\Rightarrow$ Final matrix 3×1

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$P(x) = a_0 + a_1 x + a_2 x^2 = 2 + \frac{1}{2}x + \frac{1}{2}x^2$$

$x_3 = 2.70$

y_3 ??

$$P(2.70) = 2 + \frac{2.7}{2} + \frac{2.7^2}{2}$$

$$= P(2.7) = \frac{1}{2}(4 + 2.7 + 7.29) = 6.995$$

y_3

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Derive an expression of a polynomial for the following data,

Example 2.4 *The following values of the function $f(x) = \sin x + \cos x$, are given*

x	$10°$	$20°$	$30°$
$f(x)$	1.1585	1.2817	1.3660

Find the value of $\left(\frac{\pi}{12}\right)$ and Compare the results.

Solution

Convert values of degrees in rad values

$$\boxed{\frac{x_0}{180} = \frac{x_0\,rad}{\pi}}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$x_0 = \left(\frac{10}{180}\right)\pi = 0.1745$ & $x_1 = \frac{20}{180}(\pi) = 0.3491$ & $x_2 = \frac{30}{180}(\pi) = 0.5236$

Values are approximated

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From the given table we write

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$y_0 : 1.1585 \qquad x_0 = 0.1745$

$y_1 : 1.2817 \qquad x_1 = 0.3491$

$y_2 : 1.3660 \qquad x_2 = 0.5236$

Find the value of f(x), when x=pi/12.

First Column

$$\frac{x_1x_2}{(x_2-x_0)(x_1-x_0)} = \frac{0.1828}{(0.5236-0.1745)(0.3491-0.1745)} \quad b_{11}$$

$$\frac{-(x_2+x_1)}{(x_2-x_0)(x_1-x_0)} = \frac{-(0.8728)}{(0.5236-0.1745)(0.3491-0.1745)} \quad b_{21}$$

$$\frac{1}{(x_2-x_0)(x_1-x_0)} = \frac{1}{(0.5236-0.1745)(0.3491-0.1745)} \quad b_{31}$$

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For the given table

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$\Rightarrow$ $y_0 : 1.1585$ $\quad x_0 : 0.1745$

$y_1 : 1.2817$ $\quad x_1 : 0.3491$

$y_2 : 1.3660$ $\quad x_2 : 0.5236$

b32

Second Column

$$\frac{-x_0 x_2}{(x_1-x_0)(x_2-x_1)} = \frac{-0.0914}{(0.3491-0.1745)(0.5236-0.3491)} \qquad b_{12}$$

$$\frac{(x_2+x_0)}{(x_1-x_0)(x_2-x_1)} = \frac{+(0.6981)}{(0.3491-0.1745)(0.5236-0.3491)} \qquad b_{22}$$

$$\frac{-1}{(x_1-x_0)(x_2-x_1)} = \frac{(-1)}{(0.3491-0.1745)(0.5236-0.3491)} \qquad b_{32}$$

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From the given table we write

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 x_2}{(x_2-x_0)*(x_1-x_0)} & \frac{-x_0 x_2}{(x_1-x_0)*(x_2-x_1)} & \frac{x_0 x_1}{(x_2-x_0)*(x_2-x_1)} \\ \frac{-(x_2+x_1)}{(x_2-x_0)*(x_1-x_0)} & \frac{(x_2+x_0)}{(x_1-x_0)*(x_2-x_1)} & \frac{-(x_1+x_0)}{(x_2-x_0)*(x_2-x_1)} \\ \frac{1}{(x_2-x_0)*(x_1-x_0)} & \frac{-1}{(x_1-x_0)*(x_2-x_1)} & \frac{1}{(x_2-x_0)*(x_2-x_1)} \end{pmatrix} * \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

$y_0 : 1.1585$ $\quad x_0 = 0.1745$

$y_1 : 1.2817$ $\quad x_1 = 0.3491$

$y_2 : 1.3660$ $\quad x_2 = 0.5236$

Third Column

$$\frac{x_0 x_1}{(x_2-x_0)(x_2-x_1)} = \frac{0.0609}{(0.5236-0.1745)(0.5236-0.3491)} \quad b_{13}$$

$$\frac{-(x_1+x_0)}{(x_2-x_0)(x_2-x_1)} = \frac{-0.5236}{(0.5236-0.1745)(0.5236-0.3491)} \quad b_{23}$$

$$\frac{+1}{(x_2-x_0)(x_2-x_1)} = \frac{+1}{(0.5236-0.1745)(0.5236-0.3491)} \quad b_{33}$$

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For the Coefficients a_0, a_1, a_2

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2.9967 & -2.9967 & 1 \\ -14.307 & 22.8885 & -8.5977 \\ 16.3934 & -32.7869 & 16.4204 \end{pmatrix} \times \begin{pmatrix} 1.1585 \\ 1.2817 \\ 1.366 \end{pmatrix} \begin{matrix} y_0 \\ y_1 \\ y_2 \end{matrix}$$

$$\begin{Bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{Bmatrix}$$

$a_0 = 0.9968$

$a_1 = 1.0176$

$a_2 = -0.6009$

$P(x) = a_0 + a_1 x + a_2 x^2$

Polynomial

$P(x) = 0.9968 + 1.0176x - 0.6009\, x^2$

$\frac{\pi}{12} = 0.2618$

$P\left(\frac{\pi}{12}\right) = 0.9968 + 1.0176\left(\frac{\pi}{12}\right) + (-0.6009)\left(\frac{\pi}{12}\right)^2 = 1.222$

original formula

$X_3 = \pi/12$

$\sin\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right) = 1.2247$

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