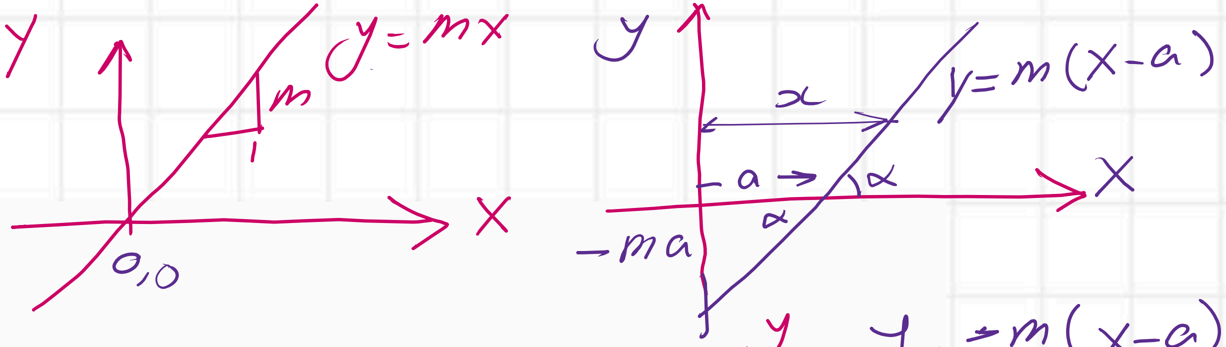
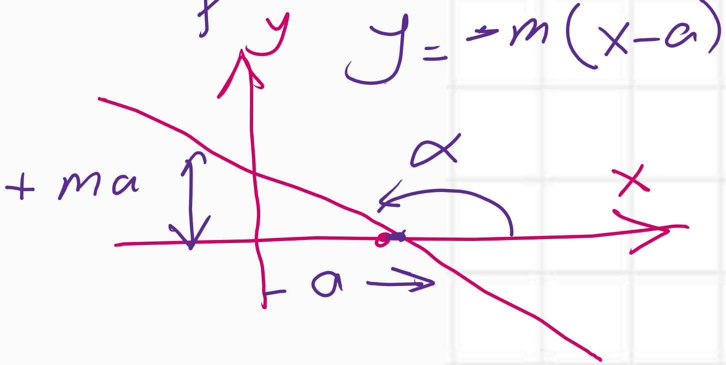
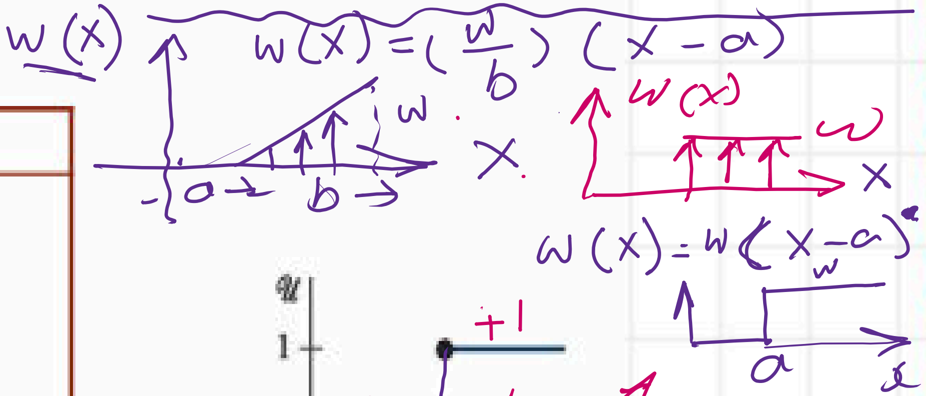
Step Functions



4.3.2 Translation on the *t*-axis



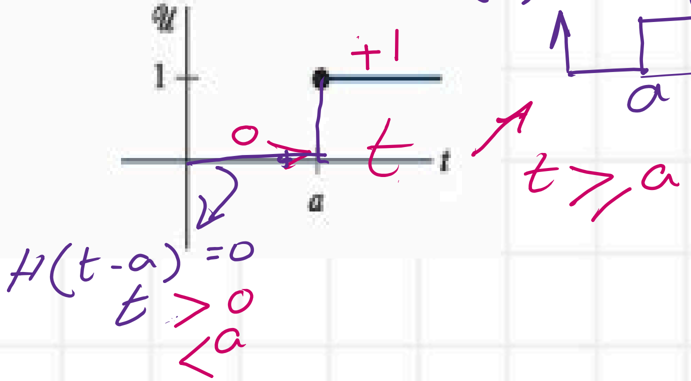
Unit Step Function In engineering, one frequently encounters functions that are either "off" or "on." For example, an external force acting on a mechanical system or a voltage impressed on a circuit can be turned off after a period of time. It is convenient, then, to define a special function that is the number 0 (off) up to a certain time $t = a$ and then the number 1 (on) after that time. This function is called the **unit step function** or the **Heaviside function** named after the renowned English electrical engineer, physicist, and mathematician **Oliver Heaviside** (1850–1925). The Heaviside layer in the ionosphere which can reflect radio waves is named in his honor.



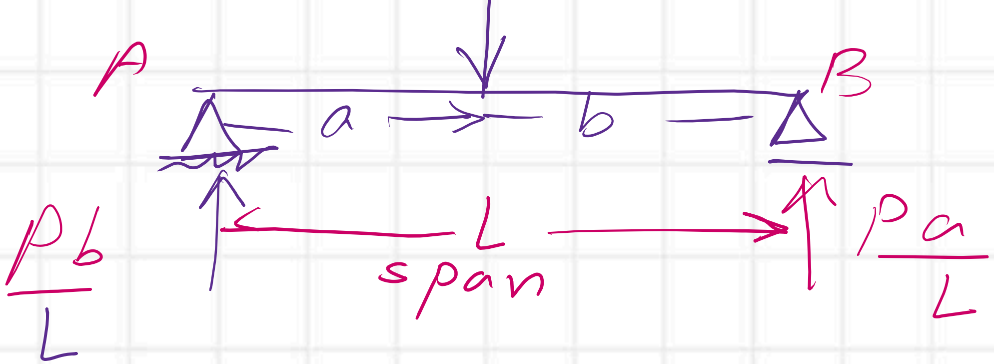
Definition 4.3.1 Unit Step Function

The **unit step function** $\mathcal{U}(t-a)$ is defined to be

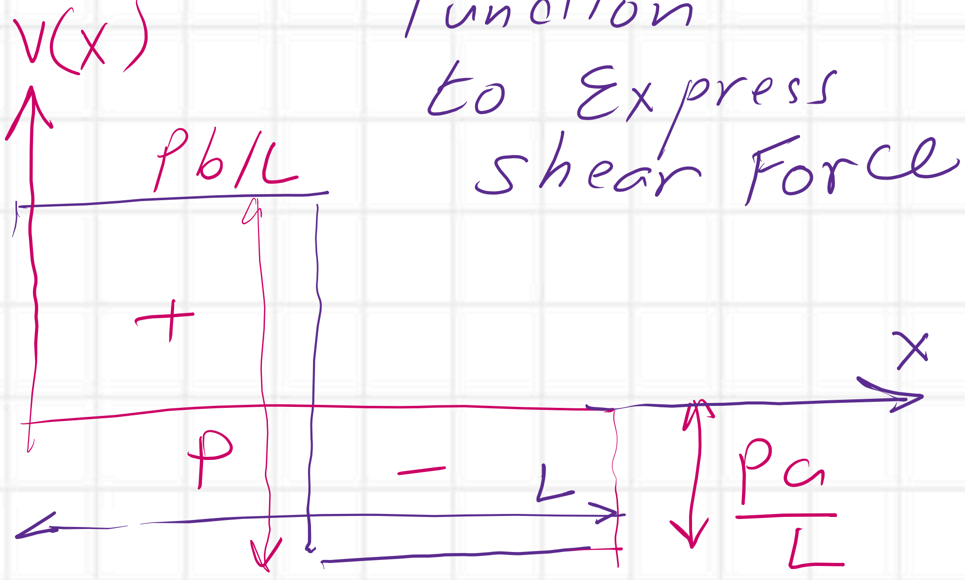
$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$



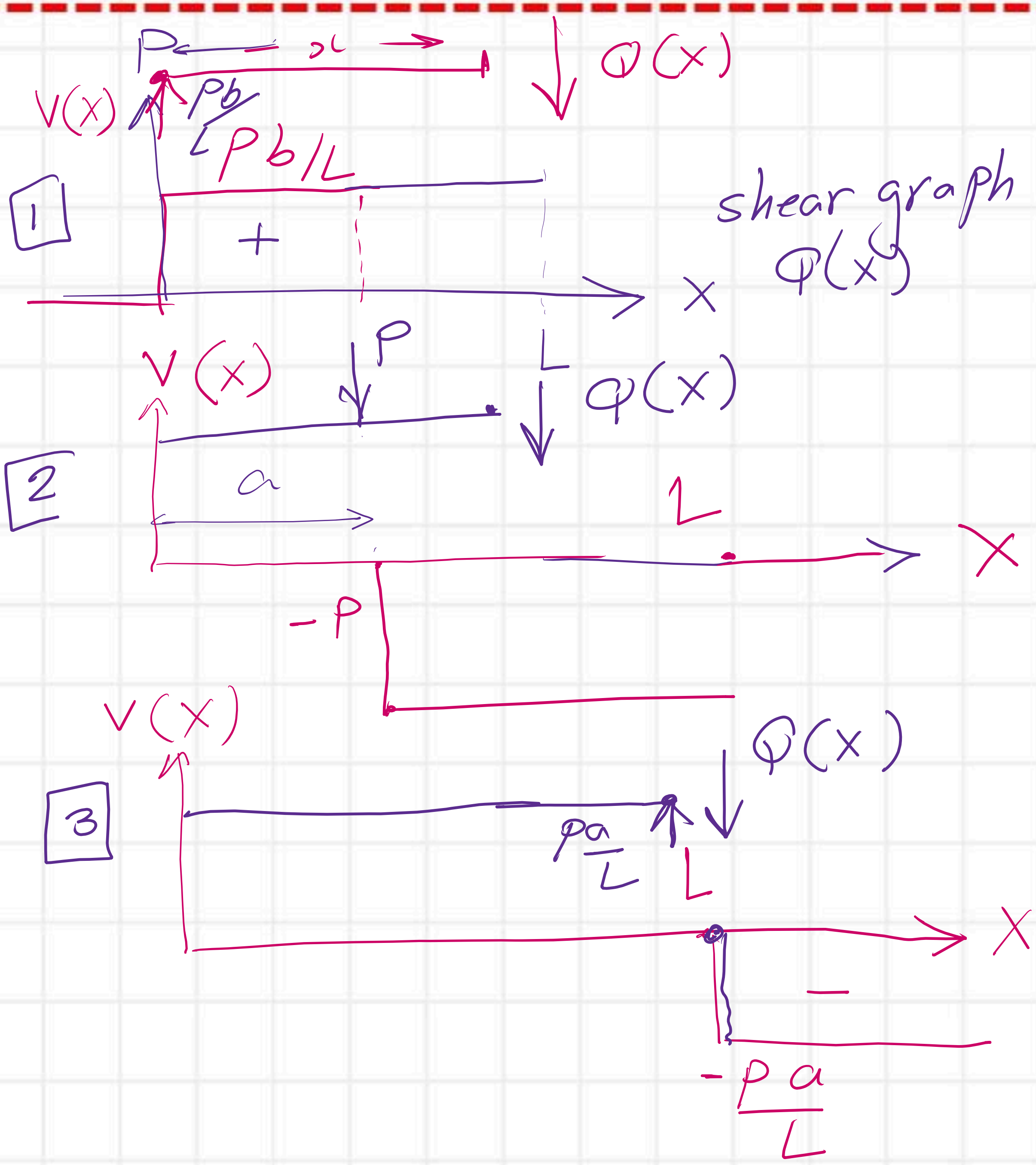
Mechanics of material



Use step Function to Express shear Force



$a + b = \ell$



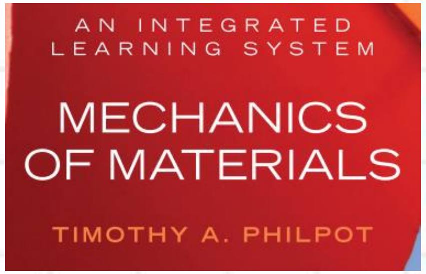
Macaulay Functions

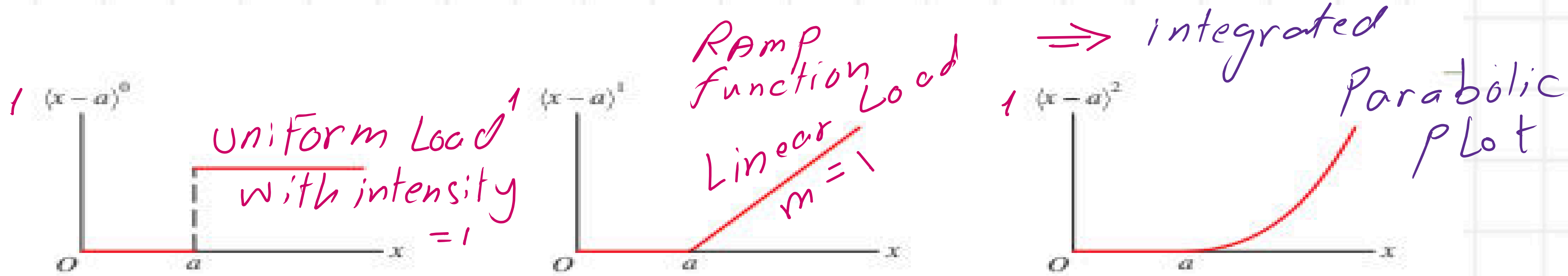
Distributed loadings can be represented by Macaulay functions, which are defined in general terms as follows:

$$\langle x - a\rangle^n = \begin{cases} 0 & \text{when } x < a \\ (x - a)^n & \text{when } x \geq a \end{cases} \quad \text{for } n \geq 0 \ (n = 0,\ 1,\ 2, \ldots) \tag{7.7}$$

Whenever the term inside the brackets is less than zero, the Macaulay function equals zero and it is as if the function does not exist. However, when the term inside the brackets is greater than or equal to zero, the Macaulay function behaves like an ordinary function, which would be written with parentheses. In other words, the Macaulay function acts like a switch in which the function turns on for values of x greater than or equal to a.

Three Macaulay functions corresponding, respectively, to $n = 0$, $n = 1$, and $n = 2$ are plotted in Figure 7.13. In Figure 7.13a, the function $\langle x - a\rangle^0$ is discontinuous at $x = a$, producing a plot in the shape of a step. Accordingly, this function is termed a **step function**.





(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

FIGURE 7.13 Graphs of Macaulay functions.

From the definition given in Equation (7.7), and with the recognition that any number raised to the zero power is defined as unity, the step function can be summarized as

$$\boxed{\langle x - a\rangle^0 = \begin{cases} 0 & \text{when } x < a \\ 1 & \text{when } x \geq a \end{cases}} \tag{7.8}$$

When scaled by a constant value equal to the load intensity, the step function $\langle x - a\rangle^0$ can be used to represent uniformly distributed loadings. In Figure 7.13*b*, the function $\langle x - a\rangle^1$ is termed a **ramp function** because it produces a linearly increasing plot beginning at $x = a$. Accordingly, the ramp function $\langle x - a\rangle^1$, combined with the appropriate load intensity, can be used to represent linearly distributed loadings. The function $\langle x - a\rangle^2$ in Figure 7.13*c* produces a parabolic plot beginning at $x = a$.

Observe that the quantity inside of the Macaulay brackets is a measure of length; therefore, it will include a length dimension, such as meters or feet. The Macaulay functions will be scaled by a constant to account for the intensity of the loading and to ensure that all terms included in the load function $w(x)$ have consistent units of force per unit length. Table 7.2 gives discontinuity expressions for various types of loads.

Singularity Functions

when $n < 0$

Singularity functions are used to represent concentrated forces P_0 and concentrated moments M_0. A concentrated force P_0 can be considered a special case of a distributed load in which an extremely large load P_0 acts over a distance ε that approaches zero (Figure 7.14*a*). Thus, the intensity of the loading is $w = P_0/\varepsilon$, and the area under the loading is equivalent to P. This can be expressed by the singularity function

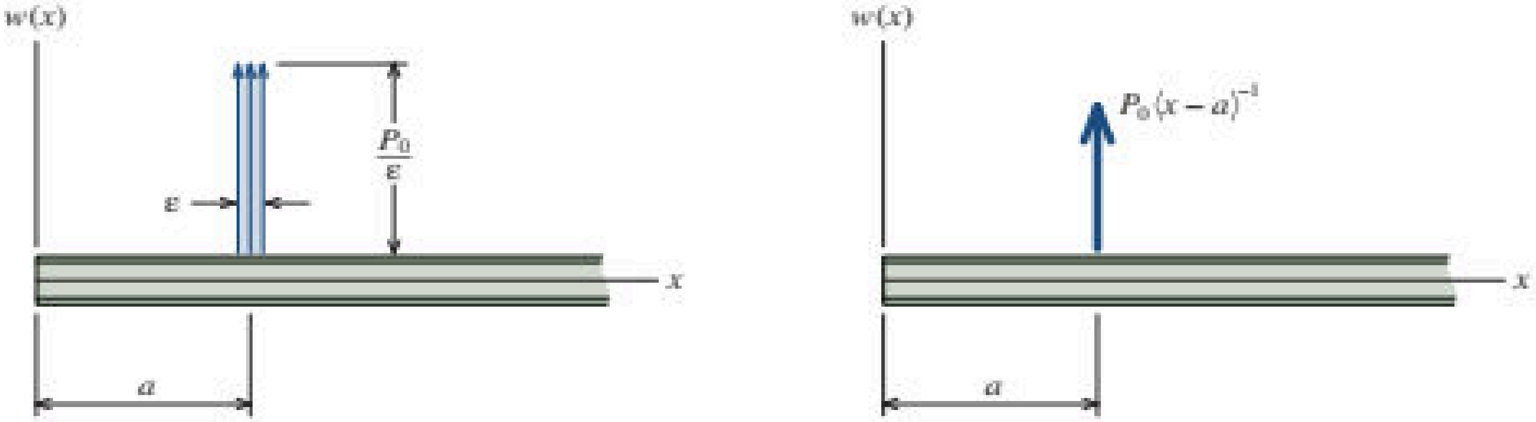
$$\boxed{w(x) = P_0 \langle x - a \rangle^{-1} = \begin{cases} 0 & \text{when } x \neq a \\ P_0 & \text{when } x = a \end{cases}} \tag{7.9}$$

in which the function has a value of P_0 only at $x = a$ and is otherwise zero. Observe that $n = -1$. Since the bracketed term has a length unit, the result of the function has units of force per unit length, as required for dimensional consistency.

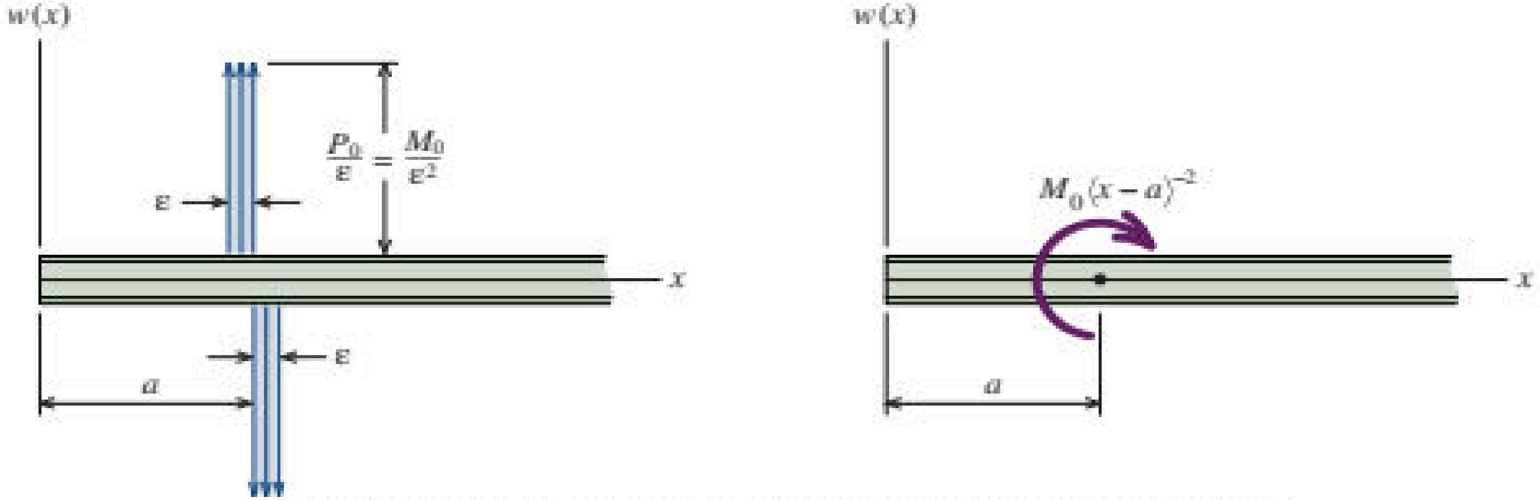
Similarly, a concentrated moment M_0 can be considered as a special case involving two distributed loadings, as shown in Figure 7.14*b*. For this type of load, the following singularity function can be employed:

$$\boxed{w(x) = M_0 \langle x - a \rangle^{-2} = \begin{cases} 0 & \text{when } x \neq a \\ M_0 & \text{when } x = a \end{cases}} \tag{7.10}$$

As before, the function has a value of M_0 only at $x = a$ and is otherwise zero. In Equation (7.10), notice that $n = -2$, which ensures that the result of the function has consistent units of force per unit length.



(*a*) Concentrated force as a special case of a distributed load



(*b*) Concentrated moment as a special case of a distributed load

FIGURE 7.14 Singularity functions to represent (*a*) concentrated forces and (*b*) concentrated moments.

Integration of Discontinuity Functions

These functions can be integrated almost like ordinary functions:

Macaulay functions ($n \geq 0$):

add 1 to n

$$\int_0^x F_n(x) = \frac{F_{n+1}(x)}{n+1} \qquad \text{i.e.} \qquad \int_0^x [x-a]^n = \frac{[x-a]^{n+1}}{n+1}$$

divide by (n+1)

Singularity functions ($n < 0$):

add 1 to n

$$\int_0^x F_n(x) = F_{n+1}(x) \qquad \text{i.e.} \qquad \int_0^x [x-a]^n = [x-a]^{n+1}$$

No divison by (n+1)