Content of Post # 3 - numerical Linear

ⓐ Illustation of how to get L & U For 2×2 matrix Using Doolittle 's method.

ⓑ How Can We Use Elementary matrix to get L & U matrices

ⓒ How Can We Solve two linear Equations Using L U ?

ⓓ Quick way to Find x, y Values For two Simultanous equations

What has happened?

From Row. echelon form.

Gauss_elimination

For an upper matrix

$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow ref$

$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \rightarrow \frac{-\frac{3}{2}R_1 + R_2}{R_2}$

U

$\begin{bmatrix} \boxed{2 \quad 3} \\ 0 \quad -\frac{1}{2} \end{bmatrix}$

no change in isf row

Where $U_{11} = a_{11}$

$U_{12} = a_{12}$

$U_{22} = a_{22} - \left(\frac{a_{21}}{a_{11}}\right)$

This is the final upper matrix

while

by defintion

$L = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix}$

$L_{21} = \frac{a_{21}}{a_{11}}$

$L = \Rightarrow \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix}$

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What has happened?

$L \quad U$

Check $A = L\,U$

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow L \cdot U = \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -1/2 \end{bmatrix}$$

Row → | Column multiplication

$$\begin{bmatrix} 1(2) + 0(0) & 1(3) + 0(-1/2) \\ \frac{3}{2}(2) + 1(0) & 3\left(\frac{3}{2}\right) + 1(-1/2) \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \rightarrow$$ we will find x, y value use A-matrix

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Use Elementary matrix to Find L & U

2x2 Matrix _ Doolittle's method

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$L_{11} = L_{22} = 1$

$L_{21} = \dfrac{a_{21}}{a_{11}}$

$U_{11} = a_{11}$

$U_{12} = a_{12}$

① First we get U

U

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \xrightarrow[R_2]{-\frac{3}{2}R_1 + R_2} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_2]{-\frac{3}{2}R_1 + R_2} \begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix} \Rightarrow E_1$$

Check $E_1 A$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{A} = \overset{U}{\begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}}$$

$U = E_1 A$ where E_1^{-1} inverse of E_1

$E_1^{-1} U = E_1^{-1} E_1 A$ $\Rightarrow$ will be L matrix

$\Rightarrow B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B^{-1} = \dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{(ad-cb)}$

$|B| = ad-cb$

$E_1^{-1} U = I A$ But $L U = A$

(2) get L as E_1^{-1}

$$L = E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix}^{-1}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ +1.5 & 1 \end{bmatrix} / (1 - 0) = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix} = L$$

$$L = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

Doolittle's method where $L_{11} = L_{22} = 1$

If we have for two set of equations : $2x+3y=13$ & $3x+4y=18$

We wont to get x & y values.

Using L U decomposition

Solution

$A * X = B \xrightarrow{as} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$

$A \cdot X = B$

Consider $L \cdot U = A \rightarrow$ substitute

then $[L\ U] X = B \rightarrow$ let $UX = C$

$L \cdot C = B$ — L & B are knowns

Multiply both sides by L^{-1}) $L^{-1} \cdot L \cdot C = L^{-1} \cdot B$

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If we have for two set of equations : 2x+3y=13 & 3x+4y=18

we want to get x, y values → get L^{-1} value

Solution

$A * X = B$ as → $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$

$A \quad X \quad = \quad B$

$L^{-1}(L)\,C = L^{-1}(B)$

From before we have

$L = \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix} \Rightarrow L^{-1}$ → swap diagonal + Chang sign Divide by $|L|$ → $L^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}$

$|L| = 1 - 0 = 1$

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If we have for two set of equations : 2x+3y=13& 3x+4y=18

we want to get x, y values $\Rightarrow$ get C Value

Solution

$A * X = B \xrightarrow{as} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$

$A \quad X \quad = B$

$C = L^{-1}(B)$

$L^{-1} \cdot B = \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix} \Rightarrow C = \begin{bmatrix} 13+0 \\ -\frac{3}{2}(13)+18 \end{bmatrix} \Rightarrow \begin{bmatrix} 13+0 \\ -19.5+18 \end{bmatrix} = \begin{bmatrix} 13 \\ -1.50 \end{bmatrix}$

But $C = U \cdot X$

$U = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$

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Remember $U \cdot X = C$

get U^{-1} value

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ -1.50 \end{bmatrix}$$

$$I \quad X = U^{-1} C$$

$$|U| = 2\left(-\tfrac{1}{2}\right) - 0 = -1 \rightarrow (U^{-1} \cdot U)(X) = (U^{-1} C)$$

$$\rightarrow U^{-1} = \begin{bmatrix} -\frac{1}{2} & -3 \\ 0 & 2 \end{bmatrix} / (-1)$$

$$\begin{bmatrix} X \\ y \end{bmatrix} = -1 \begin{bmatrix} -\frac{1}{2} & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 13 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 6.50 - 4.50 \\ 0 \quad +3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

If we have for two set of equations : 2x+3y=13 & 3x+4y=18

Our final Check

$\begin{cases} x = 2 \\ y = 3 \end{cases}$

$$\begin{array}{ccccc} & x & & y & & & \text{R.H.S} \end{array}$$

$$2(2) + 3(3) = 4 + 9 = 13 \rightarrow ok$$

$$\begin{array}{ccccc} & x & & y \end{array}$$

$$3(2) + 4(3) = 6 + 12 = 18 \rightarrow ok$$

R.H.S

Fast way to get x - y values

$$LU = A \qquad AX = B$$

substitute

$$(LU)X = B \Rightarrow$$

Multiply both sides by $U^{-1}(L^{-1}L)(UX) = U^{-1}L^{-1}B$

L.H.S

$$IX = (U^{-1}L^{-1})(B)$$

$$X = U^{-1}L^{-1}B$$

We have $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$

Elementary Matrix

$$\overset{U}{\underset{}{}} \quad U \Rightarrow \overset{A}{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}} \xrightarrow[R_2]{-\frac{3}{2}R_1 + R_2} = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_2]{-\frac{3}{2}R_1 + R_2} \overset{E_1}{\begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix}} \left.\right\} \begin{matrix} U = E_1 A \\ \underbrace{E_1^{-1}}_{L} U = A \end{matrix}$$

$E_1^{-1} = L$ or $L^{-1} = E_1$

$$U^{-1} = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}^{-1} \Rightarrow \begin{bmatrix} -\frac{1}{2} & -3 \\ 0 & 2 \end{bmatrix} / (-1)$$

Apply $U^{-1}L^{-1}B$

$$\begin{array}{cccc} & & U^{-1} & L^{-1} \\ \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1.5 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 13 \\ 18 \end{bmatrix} = \begin{bmatrix} -52+54 \\ +39-36 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$