

Two methods to solve For x_1, x_2, x_3

(a) Consider $Ax = B \Rightarrow L U x = B, U x = C$
write $LC = B$ Find L^{-1} to get C value
From $U \rightarrow$ get $U^{-1} \Rightarrow x = U^{-1}C$
 $\rightarrow$ Plenty of estimates

(b) Quick way
Multiply by $Ax = B \Rightarrow L U x = B$
 $U^{-1} L^{-1} (L) (U) x = U^{-1} L^{-1} B$
 $I x = U^{-1} L^{-1} B$

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the i th row and j th column of A . The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

First method

$$L \underbrace{U}_C X = B$$

$$L^{-1} C = L^{-1} B$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix}$$

U

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$L^{-1} = \frac{1}{|L|} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$|L| = +1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \Rightarrow = +1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix} \Rightarrow C_{11} = (-1)^{1+1} \begin{vmatrix} M_{11} \end{vmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = +1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} M_{12} \end{vmatrix} = - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = -3$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} M_{13} \end{vmatrix} = + \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = +1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} M_{21} \end{vmatrix} = - \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} M_{22} \end{vmatrix} = + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = +1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} M_{23} \end{vmatrix} = - \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix} \Rightarrow C_{31} = (-1)^4 |M_{31}|$$

$$= + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^5 |M_{32}| = - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^6 |M_{33}| = + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

For a diagonal matrix, or a (lower or upper) triangular matrix, the determinant is just the product of the diagonal elements.

$$|L| = L_{11} (L_{22}) (L_{33})$$

$$|L| = (1)(1)(1)$$

$$= +1$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

of L Determinant matrix

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$L^{-1} = \begin{bmatrix} +1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \div 1 \Rightarrow C = L^{-1}B$$

$$B = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \Rightarrow L^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \Rightarrow C = UX$$

$$U^{-1}C = X$$

We have $U \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Determinant = $+1 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = +6$

$$U^{-1} = \frac{1}{|U|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{11} = (-1)^2 |M_{11}| = + \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6$$

$$C_{12} = (-1)^3 |M_{12}| = - \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0$$

$$C_{13} = (-1)^4 |M_{13}| = + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

We have $U \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Determinant = $+1 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = +6$

$$U^{-1} = \frac{1}{|U|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{21} = (-1)^3 |M_{21}| = - \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = -6$$

$$C_{22} = (-1)^4 |M_{22}| = + \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} = 3$$

$$C_{23} = (-1)^5 |M_{23}| = - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

We have $U \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Determinant = $+1 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = +6$

$$U^{-1} = \frac{1}{|U|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{31} = (-1)^4 |M_{31}| = + \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4 - 8 = -4$$

$$C_{32} = (-1)^5 |M_{32}| = - \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = -2$$

$$C_{33} = (-1)^6 |M_{33}| = + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = +2$$

$$U^{-1} = \frac{1}{6} \begin{bmatrix} 6 & -6 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[X] = U^{-1}C = \frac{1}{6} \begin{bmatrix} 6 & -6 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$[X] = \frac{1}{6} \begin{bmatrix} 18 - 24 + 24 \\ 12 + 12 + 12 \\ 6 + 0 - 12 \end{bmatrix}$$

$$[X] = \frac{1}{6} \begin{bmatrix} 18 \\ 24 \\ -12 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\begin{aligned} X_1 &= 3 \\ X_2 &= 4 \\ X_3 &= -2 \end{aligned}$$

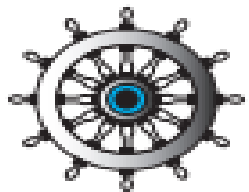
3×3
 $\xrightarrow{\quad}$
 3×1
 $\xrightarrow{\quad}$
 3×1

check $A X = B$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 + 8 - 8 \\ 9 + 32 - 28 \\ 6 + 24 - 26 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 1$

B



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

Quick solution

$$AX = B$$

$$LUX = B$$

$$(U^{-1})(L^{-1}L)(UX) = B$$

$$L^{-1}IX = U^{-1}L^{-1}B \Rightarrow X = U^{-1}L^{-1}B$$

L ⁻¹		
1	0	0
-3	1	0
1	-1	1

1/U		
1	-1	-2/3
0	1/2	-1/3
0	0	1/3

B

3 1/3	-1/3	-2/3	3
-15/6	5/6	-1/3	13
1/3	-1/3	1/3	4

x1	3
x2	4
x3	-2

3 x₁
4 x₂
-2 x₃

