Summary of Post 4

Derive an expresion for L & U for a 3X3 Matrix. Using Doolittle's method

It is possible to show that any square matrix **A** can be expressed as a product of a lower triangular matrix L and an upper triangular matrix U. The procedure based on unity elements on the major diagonal of L is called the Doolittle method. The procedure based on unity elements on the major diagonal of U is called the Crout method.

LU - Decomposition For 3x3 matrix
Doolittle 's method

Let $A \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ → while L: Lower triangular matrix

$\hookrightarrow \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$ $\underbrace{\text{Diagonals}}_{=1}$

For the upper triangular Matrix

$U \Rightarrow \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$ For $LU = A$

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LU-Decomposition For 3x3 matrix

Let $A \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow L = \underset{3\times3}{\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}}$ & $U = \underset{3\times3}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}}$

For $LU = A \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L \quad U \qquad\qquad A$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{1} & a_{32} & a_{33} \end{bmatrix}$$

$\boxed{\text{Step-1}}$ Equate

$$U_{11} = a_{11}$$
$$U_{12} = a_{12}$$
$$U_{13} = a_{13}$$

Step-2 Find L_{21} value

$$L_{21}U_{11} = a_{21}$$
$$L_{21}a_{11} = a_{21}$$
$$L_{21} = \frac{a_{21}}{a_{11}}$$
$$L_{31}U_{11} = a_{31}$$

Step-3 Find L_{31} value

$$\rightarrow L_{31} = a_{31}/U_{11} = \frac{a_{31}}{a_{11}}$$

$L \quad U$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix} = \overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}$$

$U_{11} = a_{11}$
$U_{12} = a_{12}$
$U_{13} = a_{13}$

$\Downarrow$

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Equate $L_{21}U_{12} + U_{22} = a_{22}$

$L_{21}U_{13} + U_{23} = a_{23}$

to Find U_{12} & U_{22}

$L_{21} = \frac{a_{21}}{a_{11}}$

Find

$L_{21}U_{12} + U_{22} = \frac{a_{21}}{a_{11}}(a_{12}) + U_{22} = a_{22}$

$U_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$

Find $L_{21}U_{13} + U_{23} = a_{23}$

$U_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$

$L \quad U \qquad\qquad A$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ L_{21}u_{11} & L_{21}u_{12}+u_{22} & L_{21}u_{13}+u_{23} \\ L_{31}u_{11} & L_{31}u_{12}+L_{32}u_{22} & L_{31}u_{13}+L_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$u_{11} = a_{11}$
$u_{12} = a_{12}$
$u_{13} = a_{13}$

4 $L_{21} = \dfrac{a_{21}}{a_{11}}$

$L_{31} = \dfrac{a_{31}}{a_{11}}$

$L_{11} = 1$
$L_{22} = 1$
$L_{33} = 1$

Equate $L_{31}u_{12} + L_{32}u_{22} = a_{32}$

$L_{31}u_{13} + L_{32}u_{23} + u_{33} = a_{33}$

Find L_{32}

$u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} a_{12}$

$$\frac{a_{31}}{a_{11}}(a_{12}) + L_{32}\left(a_{22} - \frac{a_{21}}{a_{11}}(a_{12})\right) = a_{32}$$

$$L_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right)}$$

Equate

$$L_{31} U_{13} + L_{32} U_{23} + U_{33} = a_{33}$$

$$L_{31} = \frac{a_{31}}{a_{11}}$$

$$U_{13} = a_{13}$$

$$U_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$$

$$L_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12} \right)}$$

$$\frac{a_{31}}{a_{11}}(a_{13}) + \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13}\right) + U_{33} = a_{33}$$

$$U_{33} = a_{33} - \frac{a_{31} a_{13}}{a_{11}} - L_{32}(U_{23}) - \frac{a_{31}}{a_{11}} a_{13}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \dfrac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{\underbrace{\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right)}_{\text{Pivot}}} & 1 \end{bmatrix}$$

This is the Lower matrix

For L_{32} we need to create U_1 matrix to get a Pivot as shown in the Following slide

$\underset{\sim}{U_1}$

$$\text{zero}\quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} -\frac{a_{21}}{a_{11}}R_1 + R_2 \\ \rightarrow \frac{a_{31}}{a_{11}}R_1 + R_3 \end{array} \Rightarrow \text{Row}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & a_{23} - \frac{a_{21}}{a_{11}}a_{13} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}}a_{12} & a_{33} - \frac{a_{31}}{a_{11}}a_{13} \end{bmatrix}$$

From the elementary Linear algebra
by Bernard Kolman - Chapter

$U_1 \longleftrightarrow L_1$ → side by side

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}-\frac{a_{21}}{a_{11}}(a_{12}) & a_{32}-\frac{a_{21}}{a_{11}}(a_{13}) \\ 0 & \left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right) & a_{33}-\frac{a_{31}}{a_{11}}a_{13} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & ? & 1 \end{bmatrix}$$

to find L_{32}

we need to make this element $= 0$, take U_{21} as pivot

Consider $\dfrac{\left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right)}{a_{22}-\frac{a_{21}}{a_{11}}a_{12}}(-1)R_2+R_3 \rightarrow R_3$ → Reverse the sign

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & & 1 \end{bmatrix}$$

L_{32}

$+\left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right) / a_{22}-a_{21}\frac{a_{12}}{a_{11}}$

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$U_1 \longleftrightarrow L_1$ side by side

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}-\frac{a_{21}}{a_{11}}(a_{12}) & a_{32}-\frac{a_{21}}{a_{11}}(a_{13}) \\ 0 & \left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right) & a_{33}-\frac{a_{31}}{a_{11}}a_{13} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & ? & 1 \end{bmatrix}$$

For U_{33} value

$$a_{33}-\frac{a_{31}}{a_{11}}a_{13}-\frac{\left[a_{33}-\frac{a_{31}}{a_{11}}a_{13}\right]}{a_{22}-\frac{a_{21}}{a_{11}}(a_{12})}\left(a_{33}-\frac{a_{31}}{a_{11}}a_{13}\right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & L_{32} & 1 \end{bmatrix}$$

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