Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

First method

$$L\underbrace{Ux}_{C} = B \qquad L^{-1}C = L^{-1}B$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix}$$

$$U \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$L^{-1} = \frac{1}{|L|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$|L| = +1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \Rightarrow = +1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix} \Rightarrow C_{11} = (-1)^{1+1} \left| M_{11} \right|$$

$$C_{11} = (-1)^{2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = +1$$

$$C_{12} = (-1)^{1+2} \left| M_{12} \right| = - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = -3$$

$$C_{13} = (-1)^{1+3} \left| M_{13} \right| = + \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = +1$$

$$C_{21} = (-1)^{2+1} \left| M_{21} \right| = - \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{22} = (-1)^{2+2} \left| M_{22} \right| = + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = +1$$

$$C_{23} = (-1)^{2+3} \left| M_{23} \right| = - \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix} \Rightarrow C_{31} = (-1)^4 \left| M_{31} \right|$$

$$= + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^5 \left| M_{32} \right| = - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^6 \left| M_{33} \right| = + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$L^{-1} = \begin{bmatrix} +1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \div 1 \Rightarrow C = L^{-1}B$$

$$B = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \Rightarrow L^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \Rightarrow C = UX$$

$$U^{-1}C = X$$

We have $U \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Determinant $= +1 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = +6$

$$U^{-1} = \frac{1}{|U|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{11} = (-1)^2 |M_{11}| = + \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 6$$

$$C_{12} = (-1)^3 |M_{12}| = - \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0$$

$$C_{13} = (-1)^4 |M_{13}| = + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

We have $U \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$$\text{Determinant} = +1 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = +6$$

$$U^{-1} = \frac{1}{|U|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{21} = (-1)^3 |M_{21}| = - \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = -6$$

$$C_{22} = (-1)^4 |M_{22}| = + \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} = 3$$

$$C_{23} = (-1)^5 |M_{23}| = - \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

We have $U \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$$\text{Determinant} = +1 \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = +6$$

$$U^{-1} = \frac{1}{|U|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{31} = (-1)^4 |M_{31}| = + \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4 - 8 = -4$$

$$C_{32} = (-1)^5 |M_{32}| = - \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = -2$$

$$C_{33} = (-1)^6 |M_{33}| = + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = +2$$

$$U^{-1} = \frac{1}{6}\begin{bmatrix} 6 & -6 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[X] = U^{-1}C = \frac{1}{6}\begin{bmatrix} 6 & -6 & -4 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$3 \times 3 \qquad 3 \times 1$$

$$3 \times 1$$

$$[X] = \frac{1}{6}\begin{bmatrix} 18 - 24 + 24 \\ 0 + 12 + 12 \\ 0 + 0 - 12 \end{bmatrix}$$

$$[X] = \frac{1}{6}\begin{bmatrix} 18 \\ 24 \\ -12 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

$x_1 = 3$

$x_2 = 4$

$x_3 = -2$

Check $AX = B$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 + 8 - 8 \\ 9 + 32 - 28 \\ 6 + 24 - 26 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

3×3 3×1

B

Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

Quick Solution

$$AX = B$$

$$LUX = B$$

$$(U^{-1})(L^{-1}L)(UX) = B$$

$$IX = U^{-1}L^{-1}B \Rightarrow X = U^{-1}L^{-1}B$$

L^{-1}

L-1

1	0	0
-3	1	0
1	-1	1

U^{-1}

1/U

1	-1	- 2/3
0	1/2	- 1/3
0	0	1/3

3 1/3	- 1/3	- 2/3
-1 5/6	5/6	- 1/3
1/3	- 1/3	1/3

B

3
13
4

x1	3
x2	4
x3	-2

3 x_1

4 x_2

-2 x_3