

Summary For the Content of Post 5A or Video Num-05A

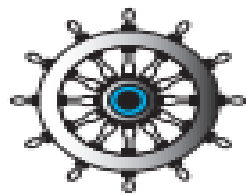
(a) Discussion do all matrices have LU
Decomposition? $\rightarrow$ sub Matrices
HELM: $\rightarrow$ Turn into I

HELM-[Helping Engineers Learn Mathematics](https://bathmash.github.io/HELM/).

<https://bathmash.github.io/HELM/>

(b) Find x_1, x_2, x_3 values for three linear equations
two methods $\rightarrow$ C Matrix

$$X = U^{-1} L^{-1} B$$



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

3. Do matrices always have an LU decomposition?

No. Sometimes it is impossible to write a matrix in the form "lower triangular" $\times$ "upper triangular".

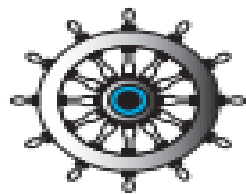
Why not?

An invertible matrix A has an LU decomposition provided that all its **leading submatrices** have non-zero determinants. The k^{th} leading submatrix of A is denoted A_k and is the $k \times k$ matrix found by looking only at the top k rows and leftmost k columns. For example if

$$|A_1| = \begin{vmatrix} 1 \end{vmatrix} = 1 \quad \text{non-zero}$$

$$|A_2| = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 1(8) - 2(3) = +2 \quad \text{non-zero}$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{vmatrix} = +6 \quad \text{non-zero}$$



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

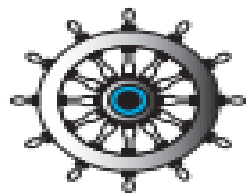
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Other way is turning matrix to I Matrix
Thru
Elementary matrices How?



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$$\begin{aligned} & \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \xrightarrow{\substack{-3R_1 + R_2 \\ -2R_1 + R_3}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{\substack{-R_2 + R_3 \\ R_3}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \\ & \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\substack{-3R_1 + R_2 \\ -2R_1 + R_3}} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \Rightarrow E_1 \quad \text{Elementary matrix } E_1 \\ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{-R_2 + R_3 \\ R_3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow E_2 \quad E_2 \end{aligned}$$



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{+R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{+R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_3+R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{R_2}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{R_3}{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_3+R_1} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_5} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc}
 \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} * \\
 E_5 & E_4 & E_3 \quad E_2
 \end{array}$$

$$\begin{array}{ccc}
 \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} & \cdot & \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \\
 E_1 & A &
 \end{array}$$

$$\begin{array}{ccc}
 \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} & \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \\
 E_4 & E_4 & E_5 \quad E_4 \quad E_3 \quad E_2 \quad E_1, A = I \\
 E_5 * E_4 & E_3 * E_2 & E_1 * A
 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$E_5 * E_4$

$E_3 * E_2$

$E_1 * A$

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{5}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 - \frac{2}{3} - \frac{4}{3} \\ 0 & 1 \\ 0 & -\frac{2}{3} + \frac{2}{3} \end{bmatrix}$$

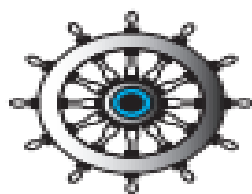
$$\begin{bmatrix} 4 - \frac{2}{3} - \frac{10}{3} \\ -\frac{5}{3} + \frac{10}{6} \end{bmatrix}$$

I

$E_5 \times E_4 \times E_3 \times E_2 \quad E_1 \times A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$
 I



Example 6

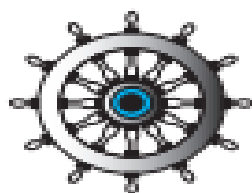
Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

(a)

Find the upper matrix
From A -matrix $AX = b$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \xrightarrow[\begin{matrix} -\frac{3}{1}R_1 + R_2 \rightarrow R_2 \\ -\frac{2}{1}R_1 + R_3 \rightarrow R_3 \end{matrix}]{U_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

to be
Zeros



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{\text{new pivot}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$- \frac{2}{2} R_2 + R_3$$

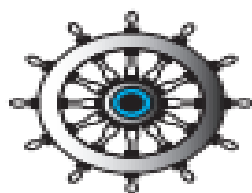
to be zero

$$\begin{aligned} a_{11} &= U_{11} \\ a_{12} &= U_{12} \\ a_{13} &= U_{13} \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

U matrix

For L_{32} Change the sign = $L_{32} = +1$
value



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

For the Lower matrix L our *Original Matrix* $\Rightarrow$ *Doolittle's method*

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & ? & 1 \end{bmatrix}$$

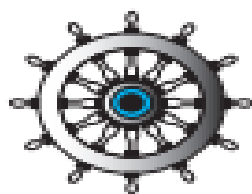
$$L_{32} = +1$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$L_{11} = L_{22} = L_{33} = 1$$

$$\frac{a_{21}}{a_{11}} = L_{21} = \frac{3}{1} = 3$$

$$\frac{a_{31}}{a_{11}} = L_{31} = \frac{2}{1} = 2$$



Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

check $LU = A$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} 3_{R11} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & (6+2) & 14 \\ 2 & (4+2) & (8+2+3) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \Rightarrow A \\ \text{Exactly } A \text{ Matrix} \end{array}$$

Two methods to solve For x_1, x_2, x_3

(a) Consider $Ax = B \Rightarrow L U x = B, Ux = C$
write $LC = B$ Find L^{-1} to get C value
From $U \rightarrow$ get $U^{-1} \Rightarrow x = U^{-1}C$
 $\rightarrow$ Plenty of estimates

(b) Quick way
Multiply by $Ax = B \Rightarrow L U x = B$
 $U^{-1} L^{-1} (L) (U) x = U^{-1} L^{-1} B$
 $I x = U^{-1} L^{-1} B$