Summary For the Content of Post 5A or Video Num-05A

@ Discussion do all matrices have LU Decomposition? → subMatrices

→ Turn into I

HELM:

HELM-Helping Engineers Learn Mathematics.

https://bathmash.github.io/HELM/

(b) Find x_1, x_2, x_3 Values For three Linear Equations

two methods → C Matrix

$$X = U^{-1} L^{-1} B$$

Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

3. Do matrices always have an LU decomposition?

No. Sometimes it is impossible to write a matrix in the form "lower triangular" × "upper triangular".

Why not?

An invertible matrix A has an LU decomposition provided that all its **leading submatrices** have non-zero determinants. The k^{th} leading submatrix of A is denoted A_k and is the $k \times k$ matrix found by looking only at the top k rows and leftmost k columns. For example if

$|A_1| = |1| = 1$ non-Zero

$|A_2| = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 1(8) - 2(3) = +2$ non-Zero

$|A_3| = \begin{vmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{vmatrix} = +6$ non_Zero

Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

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other way is turning matrix to I Matrix
Thru
Elementary matrices How?

Example 6

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U

zero

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{matrix} -3R_1+R_2 \\ -2R_1+R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \frac{-R_2+R_3}{R_3} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} -3R_1+R_2 \\ -2R_1+R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \Rightarrow E_1$$

Elementary matrix

E_1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{-R_2+R_3}{R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow E_2$$

E_2

Example 6

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$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{array}{l} \rightarrow +\frac{R_2}{2} \\ \rightarrow +\frac{R_3}{3} \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} -4R_3 + R_1 \\ \underline{-R_3 + R_2} \\ R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -2R_2 + R_1$$

E_3

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow \frac{R_2}{2} \\ \rightarrow \frac{R_3}{3} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} -4R_3 + R_1 \\ \underline{-R_3 + R_2} \\ R_2 \end{array} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_4 $\qquad$ E_5

$$\underset{E_5}{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_4}{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}} \underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} *$$

$$\underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}} =$$

$$E_5 E_4 E_3 E_2 E_1 A = I$$

$$\begin{array}{c} \\ E_4 \\ E_4 \end{array} \underset{E_5 * E_4}{\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_3 * E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}} \underset{E_1 * A}{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$E_5 * E_4 \qquad E_3 * E_2 \qquad E_1 * A$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 5/6 & -1/3 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2-\frac{2}{3}-\frac{4}{3} & 4-\frac{2}{3}-\frac{10}{3} \\ 0 & 1 & -5/3+\frac{10}{6} \\ 0 & -\frac{2}{3}+\frac{2}{3} & 1 \end{bmatrix}$$

$$\downarrow$$

$$I$$

$$I \qquad E_5 \times E_4 \times E_3 \times E_2 \quad E_1 \times A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 6

Find the solution of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the system $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$.

(a)

Find the upper matrix From A - matrix

$$A \; X = b$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \qquad \begin{matrix} -\frac{3}{1}R_1 + R_2 \longrightarrow R_2 \\ -\frac{2}{1}R_1 + R_3 \longrightarrow R_3 \end{matrix} \qquad U_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

to be Zeros

Example 6

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U_1

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

→ new pivot

$-\frac{2}{2}R_2 + R_3$

R_3

to be zero

$a_{11} = U_{11}$
$a_{12} = U_{12}$
$a_{13} = U_{13}$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

U matrix

for L_{32} change the sign value $= L_{32} = +1$

Example 6

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Doolittle's method

For the Lower matrix L

our Original Matrix $\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

$L_{11} = L_{22} = L_{33} = 1$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & ? & 1 \end{bmatrix}$$

$$\frac{a_{21}}{a_{11}} = L_{21} = \frac{3}{1} = 3$$

$$\frac{a_{31}}{a_{11}} = L_{31} = \frac{2}{1} = 2$$

$L_{32} = +1$

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$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & +1 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check $LU = A$

$$3a_{11} \quad \begin{bmatrix} 1 & 2 & 4 \\ 3 & (6+2) & 14 \\ 2 & (4+2) & (8+2+3) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \Rightarrow A$$

Exactly A Matrix

Two methods to solve for x_1, x_2, x_3

(a) Consider $Ax = B \Rightarrow LUx = B$, $Ux = C$
write $LC = B$ Find L^{-1} to get C value
From $U \rightarrow$ get $U^{-1} \Rightarrow x = U^{-1}C$
$\rightarrow$ Plenty of estimates

(b) Quick way $Ax = B \Rightarrow LUx = B$
Multiply by $U^{-1}L^{-1}(L)(U)x = U^{-1}L^{-1}B$
$Ix = U^{-1}L^{-1}B$