Post 5

3 X 3 Matrix

Doolittle's method

⓪ Find L & U Expresions for a given A Matrix.

* First method using Expresions for L_{21}, L_{31}, L_{32} & $U_{11}, U_{12}, U_{13} \rightarrow U_{23}, U_{33}$
* 2nd method Using Elementary matrices

HELM (Helping Engineers Learn Mathematics)

Task

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

LU Decomposition 30.3

3. Do matrices always have an LU decomposition?

No. Sometimes it is impossible to write a matrix in the form "lower triangular" $\times$ "upper triangular".

Why not?

An invertible matrix A has an LU decomposition provided that all its **leading submatrices** have non-zero determinants. The k^{th} leading submatrix of A is denoted A_k and is the $k \times k$ matrix found by looking only at the top k rows and leftmost k columns. For example if

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Task

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

$\rightarrow A_1$

$\rightarrow A_2$

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LU Decomposition

30.3

$A_1 = a_{11} = +3$ is not zero

$A_2 = \begin{bmatrix} 3 & 1 \\ -6 & 0 \end{bmatrix}$

(2x2) matrix

$|A_2| = 0 + 6 = +6$ not zero

$A_3 = A = \begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$

3x3 Matrix

$|A| = -6$ not zero

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![Task] Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

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LU Decomposition 30.3

Solution

Since we have checked that matrix A Can be decomposed to LU → we can proceed as follows:

↘ Factored

Our Matrix

$$\begin{array}{c} a_{11} \\ \left[\begin{array}{ccc} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{array}\right] \end{array}$$

Use a_{11} as a pivot

$$\left.\begin{array}{l} L_{21} = +\dfrac{a_{21}}{a_{11}} = \dfrac{-6}{3} = -2 \\ L_{31} = +\dfrac{a_{31}}{a_{11}} = 0/3 = 0 \end{array}\right\}$$

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Task: Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

Lower matrix where $L_{11} = L_{22} = L_{33} = 1$

we have $L_{21} = -2 \rightarrow -L_{21} = +2$

$L_{31} = 0 \qquad -L_{31} = 0$

our First Lower matrix $\qquad$ use as a multiplier U_1

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & ? & 1 \end{bmatrix}}_{L_1} \qquad \underbrace{\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}}_{A} \rightarrow \begin{array}{l} \underline{+2R_1 + R_2} \\ \quad R_2 \\ \underline{0(R_1) + R_3} \\ \quad R_3 \end{array} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix}$$

$$\underline{-L_{21}R_1 + R_2} \atop R_2$$

$$\underline{-L_{31}R_1 + R_3} \atop R_3$$

Use U_{22} as a pivot

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix}$$

Find $L32 = \frac{8}{2} = 4$

We can write

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

L_2 $\qquad$ $4 \downarrow L32$

Find L Matrix

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix} \rightarrow \underset{R3}{\underline{-L_{32}R_2 + R_3}} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

Our final U Matrix

Task

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

Solution

Check $\quad L * U = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} (1)(3) & 1(1)+0 & (1)(6)+0 \\ -2(3)+1(0) & -2(1)+2 & -12-4 \\ 0(3)+0 & 0+(4)(2) & 0-16-1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \Rightarrow \boxed{A}$$

This is the $\boxed{A}$ original matrix

2nd option

Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$.

Using Elementary matrices $\Rightarrow$ Find U Matrix

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \begin{array}{l} \to -\left(\frac{-6}{3}\right)R_1+R_2 \to \\ \to -\left(\frac{0}{3}\right)R_1+R_3 \end{array} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix} -\frac{8}{2}R_2+R_3 \to R_3 \quad \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \text{U matrix}$$

$$\left.\begin{array}{l} ⓐ\ \dfrac{2R_1 + R_2}{R_2} \\[2ex] ⓑ\ \dfrac{0R_1 + R_3}{R_3} \end{array}\right\} E_1 \quad \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \overset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\overset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}} \overset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & 16 \\ 0 & 8 & -17 \end{bmatrix}} = U$$

E_2

$$ⓒ\ \frac{-4R_2 + R_3}{R_3}$$

$$E_2 \times E_1 \Rightarrow \overset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -8 & -4 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & 16 \\ 0 & 8 & -17 \end{bmatrix} = U$$

$$E_3^{-1} E_3 A = E_3^{-1} U \Rightarrow I A = \underset{\Downarrow\ L}{E_3^{-1}} U$$

Minors and Cofactors of a Square Matrix

If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

For example, if A is a 3×3 matrix, then the minors and cofactors of a_{21} and a_{22} are as shown below.

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1}M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2}M_{22} = M_{22}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

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Inverse of 3X3 matrix

From 2x2 inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{|ad-cb|}$$

$$\underset{C}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -8 & -4 & 1 \end{bmatrix}} \Rightarrow |C| = 1 \times \underbrace{\begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix}}_{M_{11}} = 1$$

Cofactor matrix $= (-1)^{i+j}$

$$\frac{Adj}{|A|}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} = +1$$

$$C_{12} = (-1)^3 \begin{vmatrix} 2 & 0 \\ -8 & 1 \end{vmatrix} = -2$$

$$C_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -8 & -4 \end{vmatrix} = +0$$

$$C_{21} = (-1)^3 \begin{vmatrix} 0 & 0 \\ -4 & 1 \end{vmatrix} = 0$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 0 \\ -8 & 1 \end{vmatrix} = +1$$

Inverse of 3X3 matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -8 & -4 & 1 \end{bmatrix} \Rightarrow |C| = 1 \times \begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} = 1$$

C

Cofactor matrix =

$(-1)^{i+j}$

$\dfrac{\text{Adj}}{|A|}$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ -8 & -4 \end{vmatrix} = +4$$

$$C_{31} = (-1)^4 \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = +1$$

Inverse

$$\frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{32} \end{bmatrix}$$

$C_{11} : 1$ | $C_{21} : 0$ | $C_{31} : 0$

$C_{12} : -2$ | $C_{22} : 1$ | $C_{32} : 0$

$C_{13} : 0$ | $C_{23} : +4$ | $C_{33} : 1$

$$|A| = +1 \Rightarrow C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = L$$

Inverse of 3 X 3

AUGmented Matrix

$A \ A^{-1} = I$

$(A^{-1} \ A) \ A^{-1} = A^{-1}$

$I \ A^{-1} = A^{-1}$

$$[C|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -8 & -4 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \underline{-2R_1 + R_2} \\ \quad R_2 \\ \underline{+8R_1 + R_3} \\ \quad R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -4 & 1 & 8 & 0 & 1 \end{array}\right] \rightarrow \frac{4R_2 + R_3}{R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & 1 \end{array}\right] \Longrightarrow L \text{ matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Lower matrix

$$U = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

upper matrix