Objective of lecture.

- Inverse of matrix by Elementary row operation-2x2 matrix.

-Use Elementary Matrices to get the inverse of a matrix-2x2 matrix.

Prepared by Eng.Maged Kamel.

This is a fun way to find the Inverse of a Matrix:

$AX = B \rightarrow (A \mid B)$ augmented Matrix

Play around with the rows (adding, multiplying or swapping) until we make Matrix **A** into the Identity Matrix **I**

$$[A \mid I] \xrightarrow{\text{"Elementary Row Operations"}} [I \mid A^{-1}]$$

And by ALSO doing the changes to an Identity Matrix it magically turns into the Inverse!

$X = A^{-1}$

The **"Elementary Row Operations"** are simple things like adding rows, multiplying and swapping ... but let's see with an example:

then $A(\underset{X}{A^{-1}}) = I$

Perform operations $A^{-1}(A) \times A^{-1} = A^{-1}$

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2x2 Matrix

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

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Elementary row operations

- **swap** rows
- **multiply** or divide each element in a a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

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Example: Find the inverse of (2×2) matrix

$$A = \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}$$

Solution:

$$\begin{array}{c} \quad A \qquad\qquad I \\ \left[\begin{array}{cc|cc} -3 & -2 & 1 & 0 \\ 7 & 4 & 0 & 1 \end{array}\right] \end{array} \xrightarrow[R_1]{-R_1/3} \left[\begin{array}{cc|cc} +1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 7 & 4 & 0 & 1 \end{array}\right] \xrightarrow[R_2]{-7R_1+R_2}$$

zero

$$\left[\begin{array}{cc|cc} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & \frac{7}{3} & 1 \end{array}\right] \xrightarrow[R_2]{-\frac{3}{2}R_2} \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & +1 & -\frac{7}{2} & -\frac{3}{2} \end{array}\right]$$

to be $=1$

$$\left[\begin{array}{cc|cc} 1 & \overset{\text{Zero}}{\tfrac{2}{3}} & -\tfrac{1}{3} & 0 \\ 0 & +1 & -\tfrac{7}{2} & -\tfrac{3}{2} \end{array}\right] \begin{array}{l} \rightarrow -\tfrac{2}{3}R_2 + R_1 \rightarrow R_1 \\ \rightarrow \text{same} \end{array} \quad \left[\begin{array}{cc|cc} 1 & 0 & 2 & +1 \\ 0 & 1 & -\tfrac{7}{2} & -\tfrac{3}{2} \end{array}\right]$$

$$\begin{array}{cc} I & A^{-1} \end{array}$$

Inverse of A

Same inverse by determinant

$$A = \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 2 & +1 \\ -\tfrac{7}{2} & -\tfrac{3}{2} \end{bmatrix}$$

For 2×2 Matrix swap 4 & -3 change sign of 7 & -2

$$|A| = [-3(4) - (-2(7))] = -12 + 14 = +2$$

Determinant

$$\rightarrow A^{-1} = \frac{Adj}{|A|} = \frac{\begin{bmatrix} 4 & 2 \\ -7 & -3 \end{bmatrix}}{2}$$

Divide by Determinant

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An **elementary matrix** is a square matrix that is **equivalent to the identity matrix** after one elementary row operation.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Type I An elementary matrix of type I is a matrix obtained by interchanging two rows of I.

E_1 : interchange rows

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

→ Change 2nd row → First row

E2 : Type II An elementary matrix of type II is a matrix obtained by multiplying a row of I by a nonzero constant.

No chang

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix} \rightarrow \text{3 times row-2}$$

instead of 1

3 Times 2nd row $\Rightarrow (3) \underset{R_2}{\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}} = \underset{E_2}{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}$

in / position

Type III An elementary matrix of type III is a matrix obtained from I by adding a multiple of one row to another row.

$$\frac{2R_1 + R_2}{R_2}$$

$$\begin{bmatrix} 2(1) & 2(0) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \end{bmatrix} + \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\underset{E_3}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2a+c & 2b+d \end{bmatrix}$$

multiplier is in the Zero position of I

An **elementary matrix** is a square matrix that is **equivalent to the identity matrix** after one elementary row operation.

For Matrix A : $\begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}$ to get the Identity matrix I, we have done the following operations

① Divide $R_1\left(-\frac{1}{3}\right)$, we Can Use an Eliminatry matrix^{R_1} to perform that operation

$-\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E_1$ in 1's position

Try now

$$\underbrace{\begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}}_{E_1} \overset{A}{\begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}} = \begin{bmatrix} +1 & \frac{2}{3} \\ 7 & 4 \end{bmatrix}$$

(b) We have multiplied and add

$$\frac{-7R_1 + R_2}{R_2} \Rightarrow \begin{pmatrix} -7 & 0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \overset{E_2}{\begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ 7 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{14}{3} + 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}$$

0's position

(c) Third operation → multiplication

$$\frac{-\frac{3}{2}R_2}{R_2} \Rightarrow \left(-\frac{3}{2}R_2\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -\frac{3}{2}\end{bmatrix}$$

1's position

$$E_3 \to \begin{bmatrix}1 & 0\\ 0 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}1 & \frac{2}{3}\\ 0 & -\frac{2}{3}\end{bmatrix} = \begin{bmatrix}1 & \frac{2}{3}\\ 0 & +1\end{bmatrix}$$

(d) Last operation : multiply and add

$$\frac{-\frac{2}{3}R_2 + R_1}{R_1} \quad \left(0 \quad -\frac{2}{3}\right) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -\frac{2}{3}\\ 0 & 1\end{bmatrix}$$

E_4

Zero's position

$$\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow I$$

$$\begin{array}{ccccc} E_4 & E_3 & E_2 & E_1 & A \\ \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} & \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix} \end{array}$$

$$\begin{array}{cccc} & & A & \\ \begin{bmatrix} 1 & 1 \\ 0 & -\frac{3}{2} \end{bmatrix} & \begin{bmatrix} -1/3 & 0 \\ 7/3 & 1 \end{bmatrix} & \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix} & I \end{array}$$

$$A^{-1} \Leftarrow \begin{bmatrix} 2 & 1 \\ 7/2 & -3/2 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We Could write $A = \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} I$$ → Reverse Numbering

$$E_4^{-1} = \begin{bmatrix} 1 & 2/3 \\ 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2/3 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 & 2/3 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -3 & 0 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I $*\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 7 & 4 \end{bmatrix}$$

A