From : Pivoting for LU Factorization
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A permutation matrix is the identity matrix with interchanged rows.
When these matrices are multiplied by another matrix, they swap the rows or columns of the matrix.
Left multiplication by a permutation matrix will result in the swapping of rows while right multiplication will swap Columns.

Prepared by Eng.Maged Kamel.

Permutation Matrix 2×2

Identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ if we multiply By } \overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \text{ matrix}$$

Then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+0 & b+0 \\ 0+c & 0+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

We will not have in change in Matrix A.

But if we want to change the rows of Matrix A For instance swap $r_1 \rightarrow R_2$ we need to use Permutation matrix

$$\begin{array}{cc} e_1 & e_2 \end{array}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{A} = \overbrace{\begin{bmatrix} c & d \\ a & b \end{bmatrix}}^{B}$$

$a_{11} = a$
$a_{21} = c$

(a) In this 2x2 Matrix to change the arrangement of row-1 → row-2 and vice versa Use Permutation matrix denoted by P

(b) Movin $a_{11} \to a_{21}$ & $a_{22} \to a_{12}$

Use P Matrix from the Left of a matric

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

move 1st frow → 2nd row and vice versa

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How Can We Change the arrangement of Columns for a 2x2 matrix ?

But if we want to change Column arrangement.

instead of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

We will multiply by Permutation Matrix

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} b & a \\ d & c \end{bmatrix}}_{\text{Matrix } A'}$$

$2! = 2$

P_{12} P_{21}

multiply from right

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$IA = A$

P_{12} $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1 related to a
1st row

2 related to d
2nd row

$1 \rightarrow a$ $1 \rightarrow d$

To change I matrix $\rightarrow$ $\begin{array}{c} P \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$

$\boxed{1 \quad 0}$ moves down $\boxed{1 \quad 0}$

$\boxed{1 \quad 0}$ moves up $\boxed{1 \quad 0}$

$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix} \Rightarrow \begin{bmatrix} c \quad d \\ \boxed{a \quad b} \end{bmatrix}$

P_{21}

a b $\downarrow$ down

moves up
c d $\uparrow$

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There are possible 6 patterns for permution $3!$

Fixed r_1

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

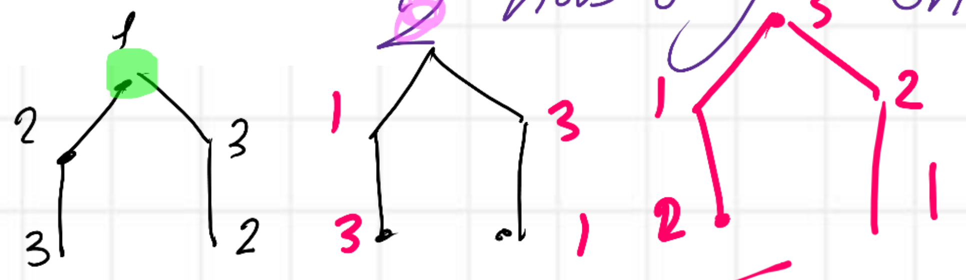
P_{123} P_{213} P_{312} P_{321} P_{132} P_{231}

Each row has only 1

Each Column has only one

From the above two examples we can observe that there are $n!$ permutation matrices of order n

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow$$ This is the identity matrix



These shapes represent ~~possible~~ permutation

We have $\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix}$ or $\begin{bmatrix}r_1\\r_3\\r_2\end{bmatrix}$

$\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}$ No change

$$\begin{bmatrix}r_2\\r_1\\r_3\end{bmatrix} \rightarrow \begin{bmatrix}r_2\\r_3\\r_1\end{bmatrix} \qquad \begin{bmatrix}r_3\\r_1\\r_2\end{bmatrix} \Rightarrow \begin{bmatrix}r_3\\r_2\\r_1\end{bmatrix}$$

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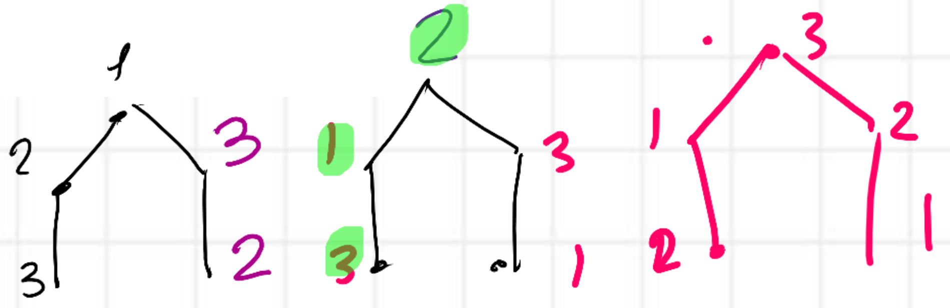
3x3 matrix has 3! = 6 Permutation matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

P_{123} P_{213} P_{312} P_{321} P_{132} P_{231}

From the above two examples we can observe that there are $n!$ permutation matrices of order n



A	B
r_1	r_2
r_2	r_1
$r_3 \rightarrow$	r_3

$P_{213} \Rightarrow$ shift row1 $\rightarrow r_2$
$r_2 \rightarrow r_1$

row-2

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\underset{A}{\begin{pmatrix}2&3&4\\1&9&3\\5&6&7\end{pmatrix}}=\underset{B}{\begin{bmatrix}1&9&3\\2&3&4\\5&6&7\end{bmatrix}}$$

row 3 no change

$(2\ 3\ 4)\downarrow$ 2nd row in Matrix B

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3x3 matrix has 3! = 6 Permutation matrices

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \rightarrow P_{231}$$

No change ← P_{123}, P_{213}, P_{312}, P_{321}, P_{132}

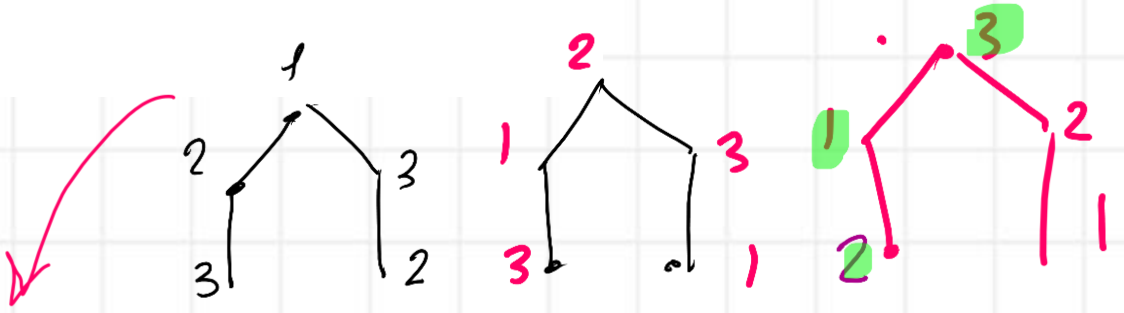
From the above two examples we can observe that there are $n!$ permutation matrices of order n

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

row-1 ↓ row-2 ↓ row-3

↑ row-1

row-3



$\boxed{P_{312}}$

A	B
r1	r3
r2 →	r1
r3 →	r2

A

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}2&3&4\\1&9&3\\5&6&7\end{pmatrix} = \begin{bmatrix}5&6&7\\2&3&4\\1&9&3\end{bmatrix}$$

(2 3 4) ↓ 2nd row in Matrix B

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3x3 matrix has 3! = 6 Permutation matrices

$$\underset{P_{123}}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}},\ \underset{P_{213}}{\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}},\ \underset{P_{312}}{\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}},\ \underset{P_{321}}{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}},\ \underset{P_{132}}{\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}},\ \underset{P_{231}}{\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}}$$

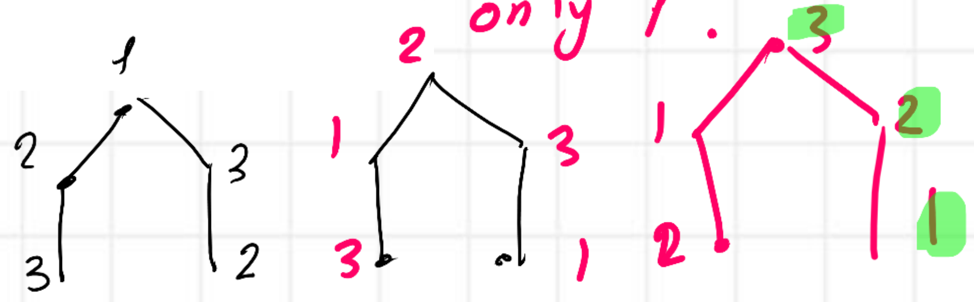
Each row has only one 1.

Each Column has only 1.

From the above two examples we can observe that there are $n!$ permutation matrices of order n

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

element 1 → 3
element 2 → 2
element 3 → 1



P_{321}

A		B
r_1	→	r_3
r_2	→	r_2
r_3	→	r_1

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\underset{A}{\begin{pmatrix}2&3&4\\1&9&3\\5&6&7\end{pmatrix}} = \underset{B}{\begin{bmatrix}5&6&7\\1&9&3\\2&3&4\end{bmatrix}}$$

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3x3 matrix has $3! = 6$ Possible arrangement

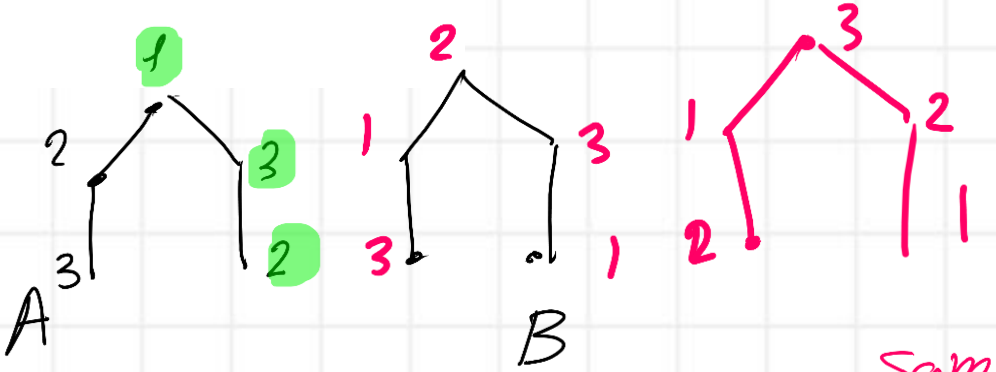
$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

P_{123} P_{213} P_{312} P_{321} P_{132} P_{231}

From the above two examples we can observe that there are $n!$ permutation matrices of order n

P_{132}

element 1 → 1
element 2 → 3
element 3 → 2



A		B
r_1	→	r_1
r_2	→	r_3
r_3	→	r_2

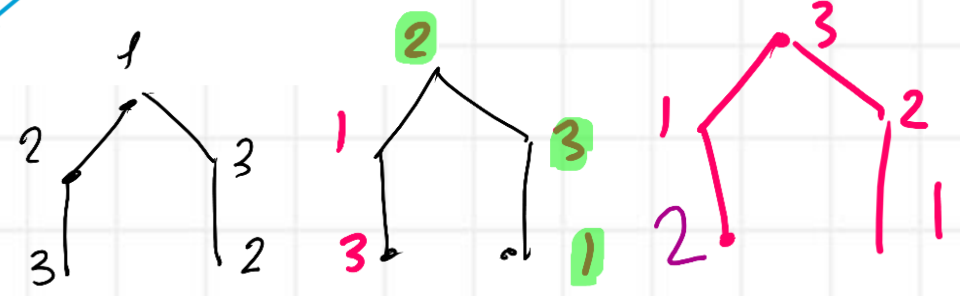
$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\underbrace{\begin{pmatrix}2&3&4\\1&9&3\\5&6&7\end{pmatrix}}_{A} = \underbrace{\begin{bmatrix}2&3&4\\5&6&7\\1&9&3\end{bmatrix}}_{B}$$

Same

3x3 matrix has 3! = 6 Permutation matrices

$$\underset{P_{123}}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}},\ \underset{P_{213}}{\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}},\ \underset{P_{312}}{\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}},\ \underset{P_{321}}{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}},\ \underset{P_{132}}{\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}},\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\ P_{231}$$

From the above two examples we can observe that there are $n!$ permutation matrices of order n



P_{231}

A	B
r_1	r_2
r_2	r_3
r_3	r_1

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\underset{A}{\begin{pmatrix}2&3&4\\1&9&3\\5&6&7\end{pmatrix}} = \underset{B}{\begin{bmatrix}1&9&3\\5&6&7\\2&3&4\end{bmatrix}}$$

$P \cdot P^T = I$ $\quad P^T$: Transpose of permutation matrix

Try

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P_{312}

3

1

2

0 0 1 → 3rd

1 0 0 → First

0 1 0 → 2nd

Then $P^{-1} = P^T$

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